\[ \prod \left\{ \frac{1}{2} \sum \gamma_{\alpha}(\varepsilon_{\alpha}(\varepsilon_{\beta} + 1) - \varepsilon_{\beta}(\varepsilon_{\gamma} + 1)) \right\} \gamma_{\beta}(\varepsilon_{\beta}(\varepsilon_{\gamma} + 1) - \varepsilon_{\gamma}(\varepsilon_{\alpha} + 1)) \gamma_{\gamma}(\varepsilon_{\gamma}(\varepsilon_{\alpha} + 1) - \varepsilon_{\alpha}(\varepsilon_{\beta} + 1)) \right\} \]

(\text{here } \alpha, \beta, \gamma \text{ are the } \beta, \gamma, \text{Pauli matrices, acting on the } \alpha \text{-th index of the spinor } \Psi \text{ and introducing an integration over the fields } \Phi, \text{ with weight } - \Delta \Phi)\), we obtain an expression for the partition function of immovable frustration lines and movable thermal vortices, which can pass only through points not occupied by frustrations but can begin and end on frustrations, with conservation of the corresponding currents. In the continuous limit, we obtain \[3.9\] for this partition function.

We once more recall that "heat capacity" is not the case in the second derivation with respect to the concentration of negative bonds (with respect to \(r\)).

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\textbf{Statistics of the levels in small metallic particles}

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1. INTRODUCTION

The principal result of quantum mechanics is the deduction that the levels of a finite system are discrete, their positions and spacing being dependent upon the boundary conditions for, and the interactions in, the system. In the simplest cases the problem of the determination of these quantities can be solved either exactly or on the basis of some approximation. But in many cases the interactions are so complicated that it is impossible to calculate the levels. At the same time, the complexity and diversity of the interactions lead to the idea of a statistical description, in which we like information about the individual levels and go over to mean quantities. Such quantities can be, for example, the density of states, the level correlations, etc. A similar approach is used in statistical physics, where information about the individual particles is given up, and averages over a large number of particles are computed.

The idea that the levels in a bounded volume can be considered statistically was first put forward by Wigner for the description of the levels in complex nuclei. In such nuclei a large number of particles interact in unknown fashion. The resulting problem consists in the exact mathematical determination of the ensemble of systems in which all the possible particle interactions are equally probable. The most convenient of the ensembles determined by Wigner is the Gaussian ensemble. According to the hypothesis on which the Gaussian statistics is based, each physical system possessing a quantum states appears in the ensemble with statistical weight \(D(N)\),

\[ D(N) = A \exp \left(-\frac{1}{2} \sum_{m} H_{mm} \right), \]

where the \(H_{nn} \) are the matrix elements of the Hamiltonian. The basic assumption used is the assumption that the various elements \(H_{nn} \) are statistically independent. This assumption is quite arbitrary. Furthermore, it is not possible to determine in terms of the \(H_{nn} \) on ensemble in which all the interactions would be equally probable.

The next step on the road to the construction of a statistical theory of levels was made by Dyson, who rejected the description with the help of the Hamiltonian matrix elements. In the Dyson theory the major role is played by an \(N \times N \) unitary matrix \(S \), where \(N \) is the number of levels. The eigenvalues of the unitary matrices \(S \) are complex numbers uniformly distributed on the unit circle. The exact correspondence between \(S \) and \(H \) is not established. According to Dyson, the behavior of a \(N(\alpha) \) successive levels of a real system is statistically identical to the behavior of \(N \) successive angles \(\theta \) on the unit circle. It is assumed that all the possible unitary matrices \(S \) appear with equal probabilities. This assumption leads to nontrivial correlation properties of the levels.

To describe centrosymmetric systems invariant under time reversal, Dyson used symmetric unitary matrices connected with an orthogonal group. Systems that are invariant under time reversal were described by arbitrary unitary matrices. Noncentrosymmetric systems that are invariant under time reversal were described by symmetric matrices. The corresponding ensembles were called orthogonal, unitary, and symplectic. It should be emphasized that the statistical description proposed by Dyson, as well as Wigner's description, is a hypothesis and not a rigorously substantiated theory. Apparently, other mathematical ensembles satisfying the same physical assumptions can be constructed.

Subsequently, such important quantities as level-correlation functions were computed on the basis of Dyson's statistical description.\(^{1,2}\) The calculations required considerable mathematical proficiency. The correlation functions for all the three ensembles were found in these calculations. In all the cases these quantities were found to be slowly decreasing oscillating functions with period corresponding to the mean level spacing. At the same time no oscillations were observed in the density of states.

In 1965 Gor'skov and Filatov applied the Dyson statistics to systems of entirely different nature. These authors considered small metallic particles. They assumed that the electrons interact not with each other, but with the impurities present in the volume. Thus, in contrast to the case

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1. sofa Phys. JETF 58 (2), August 1982
2. sofa Phys. JETF 58 (2), August 1982
of nuclei, it was not the various interparticle interactions that were random, but the external potential. The computation of the response to an external electromagnetic field requires knowledge of the level correlation function. Ref. 4 correlation functions computed on the basis of the Dyson hypothesis are used for this purpose, the orthogonal ensemble being identified with the case in which only scattering on ordinary nonmagnetic impurities occurs. For systems containing magnetic impurities, or located in a magnetic field, the unitary ensemble is used; for systems with spin-orbit interaction, the symmetric ensemble.

On the other hand, the statistical properties of the levels in metallic particles can be investigated by solving the Schrödinger equation in the given impurity potential and averaging over the positions of the impurities. Such an approach would allow us to verify the basic principles used in the phenomenological description of the levels in a bounded volume. Thus, it has not been possible to solve this problem, since the existing traditional methods, which are based on perturbation-theory-series calculations, lead to unsurmountable mathematical difficulties. At the same time, it is quite easy to formulate the problem within the framework of the supersymmetry method proposed by the present author. Below we show that the problem of the computation of the level correlation function reduces to the problem of the computation of the zero-dimensional 

\[ \sigma \text{-model introduced in Ref. 6.} \]

The symmetry group of the \( \sigma \) model depends on the presence of magnetic and spin-orbit interactions, as a result of which three different types of symmetries are possible. The zero-dimensional \( \sigma \) model calculations allow us to obtain the explicit form of the level correlation functions.

2. REDUCTION OF THE PROBLEM TO ONE IN THE ZERO-DIMENSIONAL SUPERSYMMETRY A MODEL

Let us consider a metallic particle with finite dimensions \( a \). Let us assume that the dimensions \( a \) are much greater than the interatomic distances. The quantity \( a \) can be arbitrarily large if we are considering the case of zero temperature and are investigating the lowest-lying excitations. As far as temperatures the dimension \( a \) should not exceed the critical diffusion length \( L_c \), where \( r_c \) is the inelastic-scattering time and \( D \) is the diffusion coefficient. The volume can contain randomly disposed impurities. The allowed energy values \( \epsilon_p \) are the eigenvalues for the Schrödinger equation.

\[ H = -\frac{1}{2m} \frac{\partial^2}{\partial x^2} + V(x) + \sum_{\epsilon_p} \delta(x - \epsilon_p), \]

where the \( \phi \) are the eigenfunctions corresponding to the energy \( \epsilon_p \). The angle brackets in [1] denote averaging over the positions of the impurities. In the regular part of the Hamiltonian (the kinetic energy and the energy of interaction with the external magnetic field), \( H \) is the energy of interaction with the impurities. It is assumed that the Hamiltonian \( H \) in (1) can cover both the scattering on the ordinary impurities and the scattering on the magnetic and spin-orbit impurities.

Let us define the level correlation function \( R(\omega) \) with the use of the following formula:

\[ R(\omega) = -\frac{1}{2\pi} \text{Im} \log \left( \sum_{\epsilon_p} \frac{\delta(x - \epsilon_p)}{\epsilon_p - \omega} \right) \]

where the Lagrangian \( L \) has the form

\[ L = \frac{1}{2} (\partial \phi^a)^2 - V(\phi - \lambda_a H) + \lambda_a H \phi \]

and \( \delta \phi = \lambda_a H \phi \) is a supervector having the classical Fermi and Bose variables as its components. In the case of spinless particles \( p \) has \( 2 \) components, which can be written in the form

\[ \phi = \left( \begin{array}{c} \psi \\ \chi \end{array} \right), \quad \chi = \frac{1}{\sqrt{2}} (\phi - \phi^{\dagger}) \]

The indices \( st \) denote supertransposition and complex conjugation. Upon the separation of the averages in the term \( \langle \delta \phi \rangle \phi \) in (11), the Lagrangian \( g \) goes over into

\[ L = \frac{1}{2} (\partial \phi)^2 + V(\phi) \quad \frac{1}{2} \phi^{\dagger} \phi \]

where \( SP \) is the superposition of all magnetic field. The superposition \( \langle \delta \phi \rangle \phi \) in the theory of supermatrices is the same role played by trace in the theory of ordinary matrices.

At low impurity concentrations, i.e., at concentrations such that the condition \( ra > 1 \) is fulfilled, the eigenvalues of the supermatrices \( Q \) fluctuate slowly, and can be determined by the saddle-point value of the free-energy functional corresponding to the Lagrangian (13). At the same time, the gradient to be degenerate at \( a = 0 \). The general form of the supermatrices \( Q \) corresponds to the ground state can be written as follows:

\[ Q = Q_0 + \chi (1 - Q_0) \]

where

\[ \chi = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \]

The matrices \( A, B, \) and \( \sigma \) are \( 2 \times 2 \) matrices whose elements are ordinary numbers in the cases of \( A \) and 

\[ A = \frac{1}{2} (A^T + A^T) + \chi (1 - Q_0) \]

where \( A \) and \( B \) are \( 2 \times 2 \) matrices whose elements are ordinary numbers in the cases of \( A \) and 

\[ \sigma = \frac{1}{2} (\sigma^T - \sigma^T) + \chi (1 - Q_0) \]

The strong degeneracy of the ground state at zero free energy fluctuation relates to the existence of Goldstone modes connected with the \( Q \) fluctuations. These modes and their interaction are described by a nonlinear generalized \( \sigma \) model in whose free energy the frequency plays the role of an external field:

\[ F = \frac{1}{2 \pi} \int \text{SP} \left( \frac{1}{2} \chi (1 - Q_0) - 2(\omega - \omega_0) \chi^{\dagger} \chi \right) d\Omega \]

where \( D = \sqrt{\chi^T \chi} \) is the diffusion coefficient. At sufficiently low temperatures, at which the diffusion length determined by the inelastic processes is greater than the dimensions of the system, only the states with coordinate-independent \( Q \) are important. Allowance for the higher
harmonics leads to a great increase in the energy, and can therefore be discarded. The evaluation of the integral over $\varphi$ with the Lagrangian $L$ (13), in the formula (6) is performed similarly to the computation of the density correlator in Ref. 6, and leads to the expression

$$\sum_{n} \left( \frac{G_{n}(x)}{x} \right) \left( x \frac{G_{n}(x)}{x} \right) = \int_{0}^{1} \frac{G_{n}(x)}{x} \left( 1 + \frac{G_{n}(x)}{x} \right) dx$$

in the case of potential scattering; to

$$\sum_{n} \left( \frac{G_{n}(x)}{x} \right) \left( x \frac{G_{n}(x)}{x} \right) = \int_{0}^{1} \frac{G_{n}(x)}{x} \left( 1 + \frac{G_{n}(x)}{x} \right) dx$$

in the case of magnetic interactions; and to

$$\sum_{n} \left( \frac{G_{n}(x)}{x} \right) \left( x \frac{G_{n}(x)}{x} \right) = \int_{0}^{1} \frac{G_{n}(x)}{x} \left( 1 + \frac{G_{n}(x)}{x} \right) dx$$

for $\alpha = 0$, $\beta = 0$, $\gamma = 0$, and $\delta = 0$.

To allow for the magnetic impurities and the spin-orbit interactions, we must double the dimensions of $\varphi$ and $Q$ to the case of the allowance for the spin structure. These interactions also manifest themselves in the form of effective fields that lower the symmetry of the Lagrangian (17) and from outside some of the diffusion modes. When the condition $\omega_{n} \varphi_{1}(x) + \omega_{n} \varphi_{2}(x)$ where $\omega_{n}$ and $\varphi_{n}$ are the mean free time corresponding respectively to the magnetic and spin-orbit (impurities) is fulfilled, we neglect these modes. The expressions (14), (15), and (18) remain valid in this case. As for $\psi$, the matrices $A, B, \Sigma$, and $\rho$ are given by the formula (6), in which the quantities $A_{n}, B_{n}, \Sigma_{n}$ and $\rho_{n}$ should be taken to be $2 \times 2$ unit matrices. When $\mu > 0$, $\mu < 0$, the freezing of the modes is milder. Accordingly, for the matrices $A, B, \Sigma$, and $\rho$ we have

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12} & \Sigma_{22} \end{pmatrix}, \quad \rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

in (23) and (24), the matrices $A, B, \Sigma$, and $\rho$ are $2 \times 2$ matrices.

Thus, the problem of the computation of the level correlation function for a particle with small dimensions reduces to the evaluation of the definite integral (18) over $Q$. The symmetry of the supermatrix $Q$ depends on the present magnetic and spin-orbit interactions.

Although the problem under consideration has been reduced to the definite integral (18), the evaluation of this integral is not simple because of the large number of variables, 16 for potential scattering and spin-orbit interactions. In the following section, we shall show how this calculation can be simplified by going over to "polar" matrix coordinates.
The \( \eta \) matrices coincide with the Hermitian conjugates \( \eta^* \). It must be borne in mind here that \( (\eta^*)^* = -\eta \). The \( \eta \) matrices have the same structure as the \( \gamma \) matrices.

The diagonalization of the \( \eta \) and \( \kappa \) matrices in the model with magnetic interactions leads to a situation in which only the terms linear and quadratic in \( \eta \) and \( \kappa \) remain in \( \eta \) and \( \kappa \) (31). It is not difficult to verify that in all the cases \( E_{1a} = B_{1a} = 1 \). It is no coincidence that we have written the matrices \( \eta_1 \) and \( \eta_2 \) in the form (31). The coefficients have been chosen so as to ensure the satisfaction of the equality

\[
\sum_{\alpha} |\eta_{\alpha a} - e_{\alpha a}|^2 = 0.
\]

Figuring in this equality are the products of the independent elements of the matrices \( \eta_\alpha \) and \( \kappa_\alpha \). The vanishing of these integrals allows us to regard the elements \( \eta_{\alpha a} \) and \( \kappa_{\alpha a} \) as Grassmann differentials. The matrices \( \eta_1 \) and \( \eta_2 \) can be parametrized as follows:

\[
\eta_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

The substitution of (31)-(33) into (30) enables us to write the quadratic form in terms of the independent variables \( x, \kappa, \Phi, \phi, \chi, \) and \( m, m', \eta, \). Because of the fact that the Grassmann variables \( \kappa, \Phi, \phi, \chi \) and \( \eta, m, m' \) are determined by the ordinary variables, the volume element \( d\Omega \) can be written in the form of the product

\[
d\Omega = dJ \cdot d\Omega \cdot d\theta \cdot d\phi, \tag{34}
\]

where \( dJ \) is the volume element in fermion space; \( d\Omega \cdot d\theta \cdot d\phi \), the volume element in boson space.

Constructing the Jacobians with the aid of the fermion and boson coordinate transformations, we obtain

\[
J_{\alpha 1} = \frac{1}{2} (x_0 (\eta_0 + \kappa_0) - x_0 (\eta_0 - \kappa_0)), \quad dJ_{\alpha 1} = d\eta_\alpha d\kappa_\alpha d\eta d\kappa,
\]

\[
J_{\alpha 2} = \frac{1}{2} (x_0 (\eta_0 + \kappa_0) - x_0 (\eta_0 - \kappa_0)), \quad dJ_{\alpha 2} = d\eta_\alpha d\kappa_\alpha d\eta d\kappa,
\]

\[
J_{\alpha 3} = \frac{1}{2} (x_0 (\eta_0 + \kappa_0) - x_0 (\eta_0 - \kappa_0)), \quad dJ_{\alpha 3} = d\eta_\alpha d\kappa_\alpha d\eta d\kappa,
\]

\[
J_{\alpha 4} = \frac{1}{2} (x_0 (\eta_0 + \kappa_0) - x_0 (\eta_0 - \kappa_0)), \quad dJ_{\alpha 4} = d\eta_\alpha d\kappa_\alpha d\eta d\kappa.
\]

Let us go over to the Fourier representation with the aid of the formulas

\[
I(t) = \int \sqrt{2 \pi} e^{i x t} dt.
\]

Performing the corresponding calculations for the remaining two models, i.e., for \( R_{\alpha 4} \) and \( R_{\alpha 5} \) in (36), we obtain

\[
R_{\alpha 4} (x) = \int \frac{\sin^2 x}{x^2} \sin \frac{2 \pi}{x} \sin \frac{2 \pi}{x} dt.
\]

The correlation functions \( R_{\alpha 1} (x), \) \( R_{\alpha 2} (x), \) \( R_{\alpha 3} (x), \) \( R_{\alpha 4} (x), \) \( R_{\alpha 5} (x), \) coincide exactly with the corresponding correlation functions obtained for the orthogonal, unitary, and symplectic Dyson ensembles. Let us note that this coincidence appears at the final result (42)-(44). No similarity occurs at any intermediate stage. In the case \( x \to 0 \) and \( x \to \pm \infty \), the functions \( R_{\alpha 4} (x), (42)-(44), \) have entirely different asymptotic forms:
over all the states of the system. The angle brackets in (47) now denote only averaging over the impurities. The expression (47) allows us to obtain the polarization $\alpha$:

$$\alpha = \frac{\eta^0}{\eta} \left( \frac{\eta^0}{\eta} - 1 \right)$$

where $\alpha$ is the static polarization, equal to

$$\alpha = \frac{\eta^0}{\eta} \left( \frac{\eta^0}{\eta} - 1 \right)$$

In the limit $\alpha \ll 1$ of the averaging of the matrix elements and everything else can be performed independently. Recalling the definition of the correlation function (2), and computing the matrix elements $r_r r_s$ in the same way as is done in Ref. 7, we obtain

$$\alpha = \frac{\eta^0}{\eta} \int \frac{R(k)}{\eta^0} \left( \frac{\eta^0}{\eta} - 1 \right)$$

The expression (49) establishes the connection between the two-level correlation function $R$ and the polarization in the complex frequency region. In Ref. 7 the integral (49) is evaluated, and the frequency dependences of the real and imaginary parts of the polarization are obtained. A characteristic of the expressions obtained is the fact that they preserve the oscillations when the frequency is varied.

5. CONCLUSION

The investigation carried out in the preceding sections shows that the random potential preserves the correlation between the levels in a metallic particle. The coincidence of the two-level correlation functions (42)-(44) with the results obtained for the orthogonal, unitary, and symplectic Dyson ensembles seems remarkable. At the same time we cannot assert that the formulas (42)-(44) have a universal character, and are applicable to any system in which the various types of interactions are realized. The assertion that they do not have a universal character is made, for example, in Ref. 12, where the applicability of the Dyson formulas is on the whole called into question. At the same time there exist a class of systems to which the formulas (42)-(44) are applicable. In the case of a single particle located in a random potential, it seems probable that a necessary condition for applicability of the Dyson hypothesis is the possibility of prolonged travel over the whole sample. This corresponds to the above-considered limit of long mean free times $v_{\text{rel}}$.

Unfortunately, the limit of sufficiently high impurity concentrations, where the particle is localized, and does not feel the boundaries. This leads to the disappearance of the correlations in the limit dispersions. For this same reason, apparently, the Dyson formulas are also inapplicable even in the limit of long mean free times $v_{\text{rel}}$, to a one-dimensional chain of finite length because of the localization. Let us recall that, in the limit of short mean free paths, the above-developed theory is again inapplicable to one-dimensional chains. Above we assumed that the disorder is created by the impurities located inside the sample, may be inferred that the obtained results remain valid for pure samples with surfaces of irregular form.

In the preceding sections the main attention was given to the computation of the two-level correlation function. The density of states is also an important characteristic of the system. But in the model in question this quantity is identically equal to $v$. The random potential in the limit under investigation completely smooths out all the oscillations in the density of states. Let us note that this quantity can be obtained even in the standard first-order perturbation theory. For the two-level correlation function calculation, even allowance for the diagrams the summation of which yields the diffusion modes is not sufficient. This is due to the fact that such diagrams give rise to an expansion in powers of $x^{-1}$, where $x = \nu_{\text{rel}}/v$. The terms of the type $x^{-2}$ in the oscillating functions of $x$ cannot be obtained within the framework of such a perturbation theory, since they are not analytic in $x^{-1}$. The consideration of the problem with the aid of the $\sigma$ models obtained by the replica method leads to unsurmountable mathematical difficulties due to the complexity of the computation of the integral under the constraint $Q^2 = 1$. Therefore, at present the computer procedure proposed above and based on the method of supersymmetry seems to be the only possible method of solving the formulated problem. Experimentally, the level correlation could be observed by measuring the polarization of metallic particles in weak alternating electric field. Quite a large number of recent experimental works exist. Unfortunately, quite a large spread in the particle dimensions is usually found. This leads to large fluctuations in the mean level spacing. Numerical estimates of the washing out of the oscillations for actual existing granular materials are given in Ref. 15. It is considered in that paper that the observation of the oscillations is present practically impossible. The production of granules with a small spread in the granule dimensions will be of great interest from the point of view of the observation of the correlations between the levels.

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