SOVIET PHYSICS JETP VOLUME 34, NUMBER 3 MARCH, 1972

DESTRUCTION OF LONG-RANGE ORDER IN ONE-DIMENSIONAL AND TWO-DIMENSIONAL SYSTEMS POSSESSING A CONTINUOUS SYMMETRY GROUP. II. QUANTUM SYSTEMS

V. L. BEREZINSKIĬ

Submitted February 3, 1971

Zh. Eksp. Teor. Fiz. 61, 1144–1156 (September, 1971)

The asymptotic behavior of the correlations in the low-temperature phase are found for the following two-dimensional quantum systems: a two-dimensional lattice of plane rotators, two-dimensional magnetic substances having an easy plane of magnetization, a two-dimensional Bose liquid (Sec. 1), and a two-dimensional isotropic Heisenberg ferromagnet (Sec. 2). The nature of the change of the asymptotic behavior with increasing temperature is investigated for the systems considered in Sec. 1. The nature of the change of the asymptotic behavior with increasing temperature is investigated for the systems considered in Sec. 1. The nature of the low-temperature phase consists in the superfluidity of a two-dimensional Bose liquid; the corresponding property can also be formulated in terms of magnetic substances.

IN the previous article[1] by the author, the correlation functions were determined at low temperatures in two-dimensional systems of classical spins and in other two-dimensional systems possessing a continuous symmetry group. In the present article similar results are derived for two-dimensional quantum systems: in Sec. 1 the correlations are investigated in a two-dimensional lattice of planar rotators, in a two-dimensional Bose liquid, and in planar magnetic substances (these three problems are equivalent), and in Sec. 2 the correlations in a two-dimensional Heisenberg ferromagnet are investigated. To within quantities of the order of the cutoff parameter r_0, these correlations coincide with the correlations in classical systems, which were found in[1]. The nature of the temperature-dependent corrections to the found expressions is briefly described in Sec. 1, and the question of the nature of the low-temperature phase, which was treated incorrectly in article[1], is touched upon: The characteristic of the low-temperature phase is not an infinite susceptibility, as was erroneously asserted in[1], but rather the superfluidity of a two-dimensional Bose liquid and the corresponding property of two-dimensional magnetic substances. The author hopes to return to this question, too. One-dimensional systems are not considered in this article.

1. TWO-DIMENSIONAL LATTICE OF PLANAR ROTATORS AND SYSTEMS EQUIVALENT TO IT (TWO-DIMENSIONAL BOSE LIQUID AND PLANAR MAGNETIC SUBSTANCES)

Lattice of Planar Rotators

This system corresponds to a two-dimensional lattice with lattice constant a (the notation for the lattice sites is the same as in[1]), where a planar rotator described by the dynamical variables $\hat{\varphi}_{\mathbf{r}}$ (angle) and $\hat{\nu}_{\mathbf{r}} = i\partial/\partial\varphi_{\mathbf{r}}$ (angular momentum) is associated with each site r. The wave function of the system must be periodic with respect to each of the variables $\varphi_{\mathbf{r}}$ with period 2π, and the eigenvalues $\hat{\nu}_{\mathbf{r}}$ are integers. We take the Hamiltonian of the system in the form

$$H=\sum_{\mathbf{r}}\frac{1}{2\eta}\hat{\nu}_{\mathbf{r}}^2+\frac{1}{2}\sum\sum_{|\mathbf{r}-\mathbf{r}'|=a}H_{\mathbf{r}\mathbf{r}'},\tag{1}$$

where only nearest neighbors interact ($\mathbf{r}-\mathbf{r}'=\boldsymbol{\delta}$; $|\boldsymbol{\delta}|=a$) and the interaction energy is an even periodic function of the difference $v_{\mathbf{r},\boldsymbol{\delta}}=\varphi_{\mathbf{r}'}-\varphi_{\mathbf{r}}$, having a single minimum at $v_{\mathbf{r},\boldsymbol{\delta}}=0$ in the interval $|v_{\mathbf{r},\boldsymbol{\delta}}|<\pi$, that is, the interaction is ferromagnetic:

$$H_{\mathbf{r}\mathbf{r}'}=J(v_{\mathbf{r},\delta})=\sum_{m=-\infty}^{+\infty}J_m e^{im(\varphi_{\mathbf{r}'}-\varphi_{\mathbf{r}})},\tag{2}$$

$$J(0)=\sum_{m=-\infty}^{+\infty}J_m=0,\qquad \bar{J}=J''(0)=\sum_{m=-\infty}^{+\infty}m^2J_m>0.\tag{3}$$

For the time being we shall forget about the periodicity conditions on the wave function, regarding it as a function of the variables $\varphi_{\mathbf{r}}$, which vary over the entire aixs $-\infty<\varphi_{\mathbf{r}}<+\infty$. The potential energy then has minima for configurations satisfying the conditions

$$\varphi_{\mathbf{r}'}-\varphi_{\mathbf{r}}=2\pi n_{\mathbf{r}\mathbf{r}'}\qquad(|\mathbf{r}-\mathbf{r}'|=a),\tag{4}$$

where $n_{\mathbf{r}\mathbf{r}'}$ is an arbitrary integer function, defined on neighboring pairs of sites r and r′ such that $n_{\mathbf{r}'\mathbf{r}}=-n_{\mathbf{r}\mathbf{r}'}$. Each such configuration corresponds to the minimum configuration (4). We assume that near this minimum one can replace the Hamiltonian by its quadratic expansion with respect to small deviations from (4), which gives

$$H\approx H_{(n..)}=\sum_{\mathbf{r}}\frac{1}{2\eta}\hat{\nu}_{\mathbf{r}}^2+\frac{1}{2}\sum\sum_{|\mathbf{r}-\mathbf{r}'|=a}\overset{J}{\frac{1}{2}}(\varphi_{\mathbf{r}'}-\varphi_{\mathbf{r}}-2\pi n_{\mathbf{r}\mathbf{r}'})^2.\tag{5}$$

We have a situation which is similar to the one which is encountered in the well-known method of Heitler and London, viz., in different regions of configuration space the Hamiltonian separates differently into the major part and the perturbation ("asymmetrical perturbation theory," see[2]). In the first approximation the eigenfunctions of the initial Hamiltonian will be linear combinations of the eigenfunctions of the Hamiltonians (5). In order to find these linear combinations it is, in general, necessary to solve the secular equation, but allowance for the symmetry requirements simplifies this problem. In the present case the total wave function must be periodic in all the variables $\varphi_{\mathbf{r}}$, that is, invariant with respect to the substitution $\varphi_{\mathbf{r}}\to\varphi_{\mathbf{r}}+2\pi n_{\mathbf{r}}$, where $n_{\mathbf{r}}$ is an arbitrary integer lattice function. Upon making such a substitution, $n_{\mathbf{r}\mathbf{r}'}$ in the Hamiltonian (5) goes over into another function $\bar{n}_{\mathbf{r}\mathbf{r}'}$, which is related to the previous $n_{\mathbf{r}\mathbf{r}'}$ by the equation

$$n_{\mathbf{r}\mathbf{r}'}=\bar{n}_{\mathbf{r}\mathbf{r}'}+n_{\mathbf{r}}-n_{\mathbf{r}'}.\tag{6}$$

We shall call the two functions $n_{\mathbf{r}\mathbf{r}'}$ and $\bar{n}_{\mathbf{r}\mathbf{r}'}$ $\mathscr{D}$-equivalent if they are related by equation (6) for a certain integer lattice function $n_{\mathbf{r}}$. One can easily verify that (6) is an equivalence relation and hence it divides the set of functions $n_{\mathbf{r}\mathbf{r}'}$ into nonintersecting classes of $\mathscr{D}$-equivalent functions. If $\bar{n}_{\mathbf{r}\mathbf{r}'}$ is a certain (arbitrary) representation of a given class $\mathscr{D}$, then all the others are obtained from it by formula (6), when $n_{\mathbf{r}}$ takes all possible values of the integer lattice functions. The summation over a given class $\mathscr{D}$ therefore reduces to a summation over $n_{\mathbf{r}}$. Let $\psi_{n..}^{(E)}(\ldots,\varphi_{\mathbf{r}},\ldots)$ denote the eigenfunction of (5) with energy E. The summation of all such functions for $n_{\mathbf{r}\mathbf{r}'}$, corresponding to a given class $\mathscr{D}$, has the form

$$\Psi_{\mathscr{D}}^{(E)}(\ldots,\varphi_{\mathbf{r}},\ldots)\backsim\sum_{(\ldots,n_{\mathbf{r}},\ldots)}\!\!\cdots\sum\Psi_{\bar{n}..}^{(E)}(\ldots,\varphi_{\mathbf{r}}+2\pi n_{\mathbf{r}},\ldots),\tag{7}$$

where $\bar{n}_{\mathbf{r}\mathbf{r}'}$ is a certain (arbitrary) representation of the class $\mathscr{D}$. Since (7) is an eigenfunction with the same energy E, the approximate eigenfunctions of the initial Hamiltonian must be linear combinations of the functions (7) in order to satisfy the periodicity condition.

Let us consider the physical meaning of the states described by the functions (7). Let $\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_l$ be a certain closed path (contour) along the bonds of the lattice ($\mathbf{r}_0=\mathbf{r}_l$, $\mathbf{r}_i$ and $\mathbf{r}_{i+1}$ are neighbors). Let us call the quantity $K=\sum_{i=0}^{l-1}n_{\mathbf{r}_i\mathbf{r}_{i+1}}$ the circulation of $n_{\mathbf{r}\mathbf{r}'}$ along the contour. It is easy to see that if $n_{\mathbf{r}\mathbf{r}'}$ and $\bar{n}_{\mathbf{r}\mathbf{r}'}$ are equivalent in the sense of (6), then their circulations along any arbitrary contour coincide, so that the set of circulations over all contours is an invariant of the class $\mathscr{D}$ and uniquely describes this class. It is sufficient to specify the circulation along the minimum contour—that is, along the boundaries of the elementary squares (faces) of the lattice. One can associate each face having a nonvanishing circulation with a defect, which we shall call a "vortex," since for the case of a Bose liquid (see below) these defects correspond to the well-known quantized vortices (see[3]).

Thus, each class $\mathscr{D}$ corresponds to a definite distribution of the vortices over the faces of the lattice. The states without vortices correspond to the class $\mathscr{D}_0$, having as one of its representations the function $\bar{n}_{\mathbf{r}\mathbf{r}'}=0$ (that is, consisting of functions $n_{\mathbf{r}\mathbf{r}'}$ of the form $n_{\mathbf{r}}-n_{\mathbf{r}'}$). For the class $\mathscr{D}_0$ the Hamiltonian (5) goes over into the Hamiltonian for a system of harmonic oscillators, having an energy spectrum for the low-lying excited states of the form[1)] (here and in what follows, k denotes the quasi-momentum)

$$\varepsilon_{\mathbf{k}}=\hbar\omega_{\mathbf{k}}=hc|\mathbf{k}|+o(|\mathbf{k}|),\qquad c=(a/\hbar)\sqrt{\bar{J}/\eta}.\tag{8}$$

For classes $\mathscr{D}$ different from $\mathscr{D}_0$, (5) is the Hamiltonian of a system of "displaced" harmonic oscillators and its energy spectrum is shifted with respect to (8) by an amount $\Delta E_{\mathscr{D}}$, equal to the minimum of the inhomogeneous quadratic form in (5).[2)] By virtue of this, taking the classes $\mathscr{D}\neq\mathscr{D}_0$ into consideration should give corrections of the order of $\exp(-\Delta E_{\mathscr{D}}/T)$ and at low temperatures one can confine attention to only the contributions from the class $\mathscr{D}_0$. Thereby the calculation of the averages reduces to the similar problem for a system of harmonic oscillators having the spectrum (8).

In fact, let the averaging (with a Gibbs weight function) be carried out with respect to the approximate eigenfunctions (7) of the class $\mathscr{D}_0$ and the averaged quantity is a periodic function $F_{\mathrm{per}}(\varphi_{\mathbf{r}_1},\ldots,\varphi_{\mathbf{r}_n})$ of the angles $\varphi_{\mathbf{r}_s}$. Then each term of the summation (7) gives the same contribution to the average, and it coincides with the average of $F_{\mathrm{per}}(\hat{\varphi}_{\mathbf{r}_1},\ldots,\hat{\varphi}_{\mathbf{r}_n})$ over the system of oscillators. In particular, for integer values of m_s one can write

$$\left\langle\exp\left\{i\sum_{s=1}^{n}m_s\varphi_{\mathbf{r}_s}(t_s)\right\}\right\rangle=\exp\left\{-\frac{1}{2}\sum_{s,}\sum_{s'=1}^{n}m_sm_{s'}\langle\hat{\varphi}_{\mathbf{r}_s}(t_s)\hat{\varphi}_{\mathbf{r}_{s'}}(t_{s'})\rangle_0\right\}\tag{9}$$

where $\langle\hat{\varphi}_{\mathbf{r}}(t)\hat{\varphi}_{\mathbf{r}'}(t')\rangle_0$ is the correlation function for the coordinates of the harmonic oscillators. One can easily deduce Eq. (9) for a system of harmonic oscillators for arbitrary values of m_s, but, according to what was said above, in that case when m_s is an integer and hence the averaged function is periodic, its average with respect to the system of oscillators coincides (for $T/\bar{J}\ll1$) with the Gibbs average over the initial system.

For $t_s\neq t_{s'}$ the correlation function $\langle\varphi_{\mathbf{r}_s}(t_s)\varphi_{\mathbf{r}_{s'}}(t_{s'})\rangle_0$ depends on the method of ordering the noncommuting operators $\varphi_{\mathbf{r}}(t)$, but at large distances these differences are of the order of $1/u_{ss'}$, where $u_{ss'}$ is defined as

$$u_{ss'}=\begin{cases}|\mathbf{r}_s-\mathbf{r}_{s'}| & \text{if}\quad |\mathbf{r}_s-\mathbf{r}_{s'}|>c|t_s-t_{s'}|\\ c|t_s-t_{s'}|+\sqrt{c^2(t_s-t_{s'})^2-(\mathbf{r}_s-\mathbf{r}_{s'})^2} & \text{if}\quad |\mathbf{r}_s-\mathbf{r}_{s'}|<c|t_s-t_{s'}|\end{cases}\tag{10}$$

that is, the operators $\varphi_{\mathbf{r}}(t)$ at remote points asymptotically commute.

For $u_{ss'}\gg a$ the asymptotic form of the correlators in (9) has the form

$$\langle\hat{\varphi}_{\mathbf{r}_s}(t_s)\hat{\varphi}_{\mathbf{r}_{s'}}(t_{s'})\rangle\cong O\left(\ln\frac{R}{a}\right)-(1-\delta_{s,s'})\,\alpha\ln\frac{u_{s,s'}}{r_0},\tag{11}$$

where the following notation is used ($\gamma=0.5772\ldots$ is Euler's constant)

$$\alpha=T/2\pi\bar{J},\qquad r_0=2e^{-\gamma}\hbar c/T\approx1{,}28\,\hbar c/T\tag{12}$$

and $\delta_{s,s'}$ is the Kronecker symbol ($\delta_{s,s'}=1$ for $s=s'$ and $\delta_{s,s'}=0$ for $s\neq s'$ for integer values of s, s′). The

---

[1)] The condition for the applicability of "asymmetric perturbation theory" is a small overlap of the eigenfunctions of the Hamiltonian (5) for a different $n_{\mathbf{r}\mathbf{r}'}$. For functions of the class $\mathscr{D}_0$ this reduces to $\langle(\hat{\phi}_{\mathbf{r}'}-\hat{\phi}_{\mathbf{r}})^2\rangle^{1/2}\ll1$. By calculating the standard deviation of the difference $\hat{\phi}_{\mathbf{r}'}-\hat{\phi}_{\mathbf{r}}$ from the wave function for the ground state of the system of oscillators, we obtain the condition for the validity of the method in the form $(\bar{J}\eta)^{-1/4}\gg1$.

[2)] Configurations of vortices with non-vanishing total circulation have an energy of the order of ln (R/a), where R is the size of the system, and these configurations drop out of the partition function. The minimum $\Delta E_{\mathscr{D}}=\pi^2\bar{J}$ corresponds to a pair of vortices with K = +1 and K = -1, situated in neighboring faces.

first term in (11) is a logarithmically divergent constant (for $R \to \infty$, where R is the size of the system).

Substituting (11) into (9) and taking into consideration that the first term from (11) gives a factor $\exp\{-O(\ln(R/a)(m_1 + \ldots + m_n)^2\}$ to the right-hand side of Eq. (9), which is equal to $\delta_{m_1+\ldots+m_n, 0}$ in the limit $R \to \infty$, we obtain the following result for $u_{s,s'} \gg a$:

$$\left\langle \exp\left\{ i\sum_{s=1}^{n} m_s\hat{\varphi}_{r_s}(t_s) \right\}\right\rangle \cong \delta_{m_1+\ldots+m_n, 0} \prod_{\substack{s=1 \\ (s\neq s')}}^{n} \prod_{s'=1}^{n} \left(\frac{u_{s,s'}}{r_0}\right)^{\frac{1}{2}\alpha m_s m_{s'}} \quad (m_s\text{—integers}). \qquad (13)$$

Correlations involving the participation of $\nu_{\mathbf{r}}(t) = \eta d\varphi_{\mathbf{r}}/dt$ are of the same order, $O(1/u_{ss'})$, as the terms connected with the noncommutativity of the $\varphi_{\mathbf{r}}(t)$ which were thrown away earlier, so that at large distances one can neglect them and assume

$$\langle e^{im\varphi_{\mathbf{r}_1}(t)}\nu_{\mathbf{r}_2}(t_2)\rangle \approx 0, \quad \langle \nu_{\mathbf{r}_1}(t_1)\nu_{\mathbf{r}_2}(t_2)\rangle \approx 0 \qquad (u_{12} \gg a). \qquad (14)$$

Formulas (13) and (14) enable us to calculate the asymptotic behavior for any arbitrary correlation function. For $t_s = t_{s'}$ expression (13) is equivalent to formula (21) from[1], but with a different value of the cutoff parameter r_0.3)

Two-dimensional Bose Liquid

Now let us show that the long-range correlations in a two-dimensional neutral Bose liquid can be calculated on the basis of an equivalent Hamiltonian isomorphic to (1). Let us assume that the Bose condensate only exists in the ground state (for T = 0), and for $T \neq 0$ it is destroyed by the fluctuations of the phase, a description of which we also wish to derive. We introduce the following notation: m denotes the mass of the atoms, $\bar{\rho} = 1/l^2$ is the number of particles per unit area (l is the average distance between the particles), c_s is the speed of sound, $\kappa = \partial N/\partial\mu = 1/mc_s^2$ is the compressibility in the ground state. There are two correlation lengths in the problem: the average de Broglie wavelength $\xi = \hbar/mc_s$4) and r_c—the radius of correlation for fluctuations of the phase; at T = 0 (in the ground state) $r_c = \infty$, but at low temperatures $r_c \gg \xi$ and has a macroscopic value.5) Therefore, one can separate the entire system into cells, having macroscopic dimensions a such that $\xi \ll a \ll r_c$; for simplicity we shall assume that these cells are the cells of a square lattice, and their centers coincide with the lattice sites **r**.

Let us represent the Hamiltonian of the system in the form of a sum of Hamiltonians of the individual cells and of terms describing the transitions between the cells. If at first we neglect the latter, then the Hamiltonian separates into the sum of independent Hamiltonians for the individual cells. In this connection one can construct the low-lying eigenstates as products of the eigenstates of the individual cells with quantum numbers differing slightly from the equilibrium number $N_a = \bar{\rho}a^2$. Namely, let us denote the ground state of the **r**-th cell by $|\nu_{\mathbf{r}}\rangle_{\mathbf{r}}$ when the number of particles in it is given by $N_{\mathbf{r}} = N_a + \nu_{\mathbf{r}}$; then one can associate a state $|\nu.\rangle = \Pi|\nu_{\mathbf{r}}\rangle_{\mathbf{r}}$ (the product over all cells) to each distribution $\{\ldots, \nu_{\mathbf{r}}, \ldots\} = \{\nu.\}$ of the particles over the cells, in which $N_a + \nu_{\mathbf{r}}$ particles fall into the **r**-th cell (and where $\Sigma\nu_{\mathbf{r}} = 0$).

The energy of this state is composed of the energies of the individual cells and is equal to $E(\nu.) = E_0 + \Sigma(2\kappa N_a)^{-1}\nu_{\mathbf{r}}^2 + o(\nu_{\mathbf{r}}^2)$ (the linear terms in $\nu_{\mathbf{r}}$ drop out due to the fact that $\Sigma\nu_{\mathbf{r}} = 0$). Since the quadratic terms in $\nu_{\mathbf{r}}$ are small (they are of the same order as the terms associated with transitions between the cells, which have been discarded), so the states $|\nu.\rangle$ form an almost degenerate system of states. According to perturbation theory for this case, in the first-order approximation the energies are determined as the eigenvalues of an effective Hamiltonian which is equal to $H_{eff} = \hat{P}H\hat{P}$, where $\hat{P}$ is a projection operator on a subspace spanned by the states $|\nu.\rangle$. More precisely

$$H_{eff} = E_0 + \sum_{\mathbf{r}} \frac{1}{2\kappa N_a}\hat{\nu}_{\mathbf{r}}^2 + \sum\sum_{|\mathbf{r}-\mathbf{r}'|=c} \hat{P}H_{\mathbf{r}\mathbf{r}'}\hat{P}, \qquad (15)$$

where $\hat{\nu}_{\mathbf{r}}$ is the operator multiplying the state $|\nu.\rangle$ by the number $\nu_{\mathbf{r}}$, and $\hat{P}H_{\mathbf{r}\mathbf{r}'}\hat{P}$ is the term in the Hamiltonian corresponding to transitions from cell **r** to cell **r**′, from which only the matrix elements $\langle\nu.'|H_{\mathbf{r}\mathbf{r}'}|\nu.\rangle$ are left. It is obvious that these matrix elements will differ from zero only for transitions $\{\nu.\} \to \{\nu.'\}$ such that ν particles go from cell **r** to the neighboring cell **r**′ but the number of particles in the remaining cells does not change. Let us denote the value of such a matrix element by J_ν; then one can write the operators $\hat{P}H_{\mathbf{r}\mathbf{r}'}\hat{P}$ in the form

$$\hat{P}H_{\mathbf{r}\mathbf{r}'}\hat{P} = \sum_{\nu=-\infty}^{+\infty} J_\nu(\hat{R}_{\mathbf{r}})^{\nu}(\hat{R}_{\mathbf{r}'})^{-\nu}, \qquad (16)$$

where $(\hat{R}_{\mathbf{r}})^{\nu}$ denotes the operator which changes the number of particles in the **r**-th cell by ν, leaving the number of particles in the other cells unchanged.

In order to see the equivalence of the Hamiltonians (1) and (15), let us pass to a representation where the operators $\hat{R}_{\mathbf{r}}$ are diagonal. Let $|\varphi\rangle_{\mathbf{r}} = \Sigma e^{i\varphi\nu}|\nu\rangle_{\mathbf{r}}$ (the summation is over the integers ν) denote the state of the **r**-th cell containing an indeterminate number of particles, but the phase φ of the condensate is fixed. If a state $|\varphi.\rangle = \Pi|\varphi_{\mathbf{r}}\rangle_{\mathbf{r}}$ is associated with each set of phases $\{\ldots, \varphi_{\mathbf{r}}, \ldots\} = \{\varphi.\}$, then the two sets of basis functions $|\varphi.\rangle$ and $|\nu.\rangle$ are unitarily equivalent, and since

$$\hat{\nu}_{\mathbf{r}}|\varphi.\rangle = -i\frac{\partial}{\partial\varphi_{\mathbf{r}}}|\varphi.\rangle, \qquad (\hat{R}_{\mathbf{r}})^{\nu}|\varphi.\rangle = e^{i\nu\varphi_{\mathbf{r}}}|\varphi.\rangle, \qquad (17)$$

then (15) actually coincides with (1) for $\eta = \kappa N_a$. The quantity corresponding to $\bar{J}$ (see Eq. (3)) can be expressed in terms of the parameters of a Bose liquid by using the f-sum rule

$${}_0\langle N|[[\hat{\rho}_{\mathbf{k}}, H]\hat{\rho}_{-\mathbf{k}}]|N\rangle_0 = \frac{N}{m}\hbar^2\mathbf{k}^2$$

(see[4,5]), where $\hat{\rho}_{\mathbf{k}} \approx \sum_{\mathbf{r}}(N_a + \hat{\nu}_{\mathbf{r}})e^{i\mathbf{k}\cdot\mathbf{r}}$ are the Fourier components of the density, $|N\rangle_0 \approx (\text{const})\sum|\nu.\rangle$ is the

ground state (the summation extends over all $\nu_{\mathbf{r}}$ such that $-(\Delta N_a) \leq \nu_{\mathbf{r}} \leq (\Delta N_a)$, where $\Delta N_a = O(\sqrt{N_a})$ denotes the fluctuations of the number of particles in a cell). This gives $\bar{J} = \bar{\rho}\hbar^2/m$, so that the parameters appearing in (13) are given by

$$\alpha = \frac{Tm}{2\pi\bar{\rho}\hbar^2}, \quad c = \sqrt{\frac{1}{m\kappa}} = c_{зв}, \quad r_0 \cong 1{,}28\frac{\hbar c_{зв}}{T}. \qquad (18)$$

For these values of the parameters, formulas (13) and (14) permit us to determine the asymptotic form of the average of any function of $\hat{\varphi}_{\mathbf{r}}$ and $\hat{\nu}_{\mathbf{r}}$. It is necessary to associate the operators $\hat{P}\hat{A}\hat{P}$ with the quantities $\hat{A}$ of the initial system; if $\hat{A}_s$ is a local operator pertaining to the neighborhood of the point $\mathbf{r}_s$, then $\hat{P}\hat{A}_s\hat{P}$ is expressed in terms of $\hat{\varphi}_{\mathbf{r}_s}$ and $\hat{\nu}_{\mathbf{r}_s}$, and since by virtue of (14) one can set $\nu_{\mathbf{r}} \approx 0$ in evaluating the correlations at widely separated points, then the averages of the products $\prod_s \hat{A}_s$ in the initial system coincide (for widely separated points $\mathbf{r}_s$) with the averages of products of the operators $\mathscr{A}_s(\hat{\varphi}_{\mathbf{r}_s})$, where $\mathscr{A}_s(\varphi)$ is the Bogolyubov quasi-average of the operator $\hat{A}_s$ (see[6]), expressed as a function of the phase φ of the Bose condensate, that is, $\hat{A}_s \to \mathscr{A}_s(\hat{\varphi}_{\mathbf{r}_s})$, where $\mathscr{A}_s(\varphi)$ is defined as

$$\mathscr{A}_s(\varphi) = \lim_{N\to\infty}\sum_{\nu=-\infty}^{+\infty} e^{i\nu\varphi}\langle N|\hat{A}_s|N+\nu\rangle = \langle\varphi|A_s|\varphi\rangle. \qquad (19)$$

In particular, according to this rule the operator $\psi(\mathbf{r}, t)$ corresponds to the expression $\sqrt{\rho_0}\exp(i\varphi_{\mathbf{r}}(t))$, where ρ_0 denotes the density of the Bose condensate; for the asymptotic form of the average $\langle\psi^+(\mathbf{r}, t)\psi(\mathbf{r}', t')\rangle$ (for $t = t'$, when $u_{\mathbf{r},t;\mathbf{r}',t'} = |\mathbf{r} - \mathbf{r}'|$) we have

$$\langle\psi^+(\mathbf{r})\psi(\mathbf{r}')\rangle \cong \rho_0|(\mathbf{r} - \mathbf{r}')/r_0|^{-\alpha} \quad (|\mathbf{r} - \mathbf{r}'| \gg \xi). \qquad (20)$$

This formula was previously derived in[7] and[8].

Anisotropic Planar Magnetic Substances

These systems are considered in[9] where, in particular, their equivalence with Bose systems is mentioned. In the notation of[9] the rules giving the correspondences between the parameters, similar to (18), have the form: $\alpha = T/2\pi\rho_s$ and $c = (\rho_s/\chi_s)^{1/2}$ (the values of ρ_s and χ_s are taken for T = 0).

The Nature of the Corrections to (13) and the Nature of the Low-temperature Phase

The following approximations were made in the derivation of the asymptotic expressions (13) for the correlation functions: In the first place, the contributions from classes $\mathscr{D} \neq \mathscr{D}_0$ were neglected, i.e., quantized vortices were not considered; in the second place, in calculating the contributions from the class $\mathscr{D}_0$ the quadratic expansion of the Hamiltonian near the minimum was used. It is not difficult to take the corrections associated with the breakdown of the latter approximation into account by using perturbation theory; here the usual diagram technique is applicable, in which the lines correspond to the correlation functions $\langle\varphi_{\mathbf{r}}(t)\varphi_{\mathbf{r}'}(t')\rangle_0$ for a system of harmonic oscillators, and the vertices correspond to terms of the fourth-order and higher orders in the expansion of the Hamiltonian in powers of $v_{\mathbf{r},\boldsymbol{\delta}}$. Since the factors $v_{\mathbf{k},\boldsymbol{\delta}} = i(\mathbf{k}\cdot\boldsymbol{\delta})\varphi_{\mathbf{k}} + o(\mathbf{k})$ (the Fourier transforms of $v_{\mathbf{r},\boldsymbol{\delta}}$) appear in the vertices, then as $\mathbf{k} \to 0$ the "polarization operator" $\Pi(\mathbf{k}) \sim k^2$; if ρ_s^0 is introduced according to the equation

$$\lim_{\mathbf{k}\to 0}\Pi(\mathbf{k})/k^2 = (\bar{J} - \rho_s^{\,0})/T,$$

then as $\mathbf{k} \to 0$ the Fourier transform of the total correlation function $\langle\varphi_{\mathbf{k}}\varphi_{-\mathbf{k}}\rangle \to (T/\rho_s^0)k^{-2}$; from here follow (11) and (13) with $\alpha = T/2\pi\rho_s^0$. For example, for the two-point correlation function one will have

$$\langle\exp\{i(\varphi_{\mathbf{r}} - \varphi_{\mathbf{r}'})\}\rangle \cong C(T)|\mathbf{r} - \mathbf{r}'|^{-\alpha(T)} \quad (|\mathbf{r} - \mathbf{r}'| \gg a). \qquad (21)$$

Let us consider one more quantity $\Delta\mathscr{F}(A)$—the change of the free energy due to the influence of an external vector potential $A_{\mathbf{r},\boldsymbol{\delta}}$ (the corresponding Hamiltonian is obtained from (1) by making the substituting $v_{\mathbf{r},\boldsymbol{\delta}} \to v_{\mathbf{r},\boldsymbol{\delta}} + A_{\mathbf{r},\boldsymbol{\delta}}$). Let $A_{\mathbf{r},\boldsymbol{\delta}} = (\mathbf{A}(\mathbf{r})\cdot\boldsymbol{\delta})$, where $\mathbf{A}(\mathbf{r}) = \{A_i(\mathbf{r})\}$ is a slowly varying vector function of the coordinates. The diagram technique described above gives

$$\Delta\mathscr{F}(A) = \frac{1}{2}\rho_s\sum_{i,}\sum_{i'=1,2}\int (A_{\mathbf{k}})_i(A_{-\mathbf{k}})_{i'}\left(\delta_{ii'} - \frac{k_i k_{i'}}{k^2}\right)(d\mathbf{k}) + o(A^2), \qquad (22)$$

where $\rho_s = \rho_s^0$ and $(A_{\mathbf{k}})_i$ is the Fourier transform of the function $A_i(\mathbf{r})$.

One can clarify the role of the corrections associated with the classes $\mathscr{D} \neq \mathscr{D}_0$ (i.e., with the vortices) in the example of a classical system (with the same Hamiltonian (1)). For this system one is able to construct a systematic low-temperature expansion, which the author proposes to describe in a separate article. We shall, however, discuss some of the conclusions here. (It is clear that the behavior of the correlation functions at large distances is identical in the quantum and in the classical systems; we prove this below in Sec. 2 (see Eq. (34)). It turns out that the vortices are equivalent to the particles of a certain lattice gas (on the dual lattice, whose sites $\mathbf{r}_*$ lie at the centers of the "faces" of the initial lattice). The states of the gas are specified by distributions of the "charges" $K_{\mathbf{r}*}$ (the vortex circulations), where each $K_{\mathbf{r}*}$ independently takes integer values 0, ±1, ±2, ..., and the energies of the states are given by

$$E(\ldots, K_{r_*}, \ldots) = 4\pi^2\rho_s^{\,0}\Sigma\Sigma K_{\mathbf{r}_*}K_{\mathbf{r}_*'}G_{\mathbf{r}_*\mathbf{r}_*'}$$

(the sum goes over all pairs $\mathbf{r}_*, \mathbf{r}_*'$), where $G_{\mathbf{r}_*\mathbf{r}_*'} \approx 2\pi^{-1}\ln(R/|\mathbf{r}_* - \mathbf{r}_*'|)$ for $|\mathbf{r}_* - \mathbf{r}_*'| \gg a$ (R denotes the size of the lattice). One can prove that in such a gas at low temperatures all of the particles are bound into neutral "quasi-molecules" (i.e., groups of vortices having a total circulation equal to zero), and the decay of the latter is related to a certain phase transition (the situation does not resemble a gas with a Coulomb interaction $\sim |\mathbf{r}_* - \mathbf{r}_*'|^{-1}$, where there is a non-vanishing concentration of ions at any arbitrary temperature!). Upon taking the vortices into account we again have (21) and (22), where

$$\alpha = \frac{1}{2\pi}\left(\frac{T}{\rho_s^{\,0}} + 4\pi^2\overline{d^2}\right) \qquad \rho_s = \rho_s^{\,0}\left(1 - \frac{4\pi^2\overline{d^2}}{T}\rho_s^{\,0}\right) \qquad (23)$$

(the mean-square "dipole moment of the molecules" is denoted by $\overline{d^2}$). The phase transition occurs at a temperature T_c such that

---

3) In the classical case the cutoff is due to the short-wavelength behavior of $\omega_{\mathbf{k}'}$ and $r_0 \approx a$. In the quantum case the cutoff is due to the Planck distribution for $\hbar\omega_{\mathbf{k}} \approx T$, $r_0 \approx \hbar\omega_{\mathbf{k}}/T \gg a$.

4) Fulfillment of the condition of quantum degeneracy, $\xi \gg l$, is necessary for the very existence of the Bose liquid.

5) As we shall see below (see Eq. (20)), $r_c/\xi \approx (\kappa T)^{-1}\exp\{-O(\xi^2/l^2\kappa T)\}$.

$$\rho_s(T_c) = 0. \tag{24}$$

In addition, it is evident in Eq. (21) that $C(T) \to 0$ as $T \to T_c$.

The author of[1] justified the necessity of a phase transition by the fact that at low temperatures the susceptibility $\chi = (\partial\langle m\rangle/\partial h)_{h=0}$ is infinite (the average moment $\langle m\rangle$ in the field h is given by formula (40), see below). The transition to the usual dependence $\langle m\rangle \sim h$ occurs when $\alpha = 2$. Although the value of $\alpha(T_c)$ is unknown, by taking the interaction with the external field in the form $h \cos m\varphi_{\mathbf{r}}$, we obtain a similar transition when $\alpha = 2/m^2$ (where m is an arbitrary integer). These transitions pertain to the system with $h \neq 0$, but not to the initial system with $h = 0$, and an infinite susceptibility cannot be regarded as an indication of a separate phase (see footnote[6] below).

The true nature of the low-temperature phase (as the author now understands it) consists in the fact that a two-dimensional Bose liquid is superfluid; the corresponding property can also be formulated in terms of magnetic substances. In this connection the quantity ρ_s from (22) corresponds to the density of the superfluid component in the two-fluid hydrodynamics of the Bose liquid or the "coefficient of rigidity" for magnetic substances, which was introduced in[9].

Formula (22) and the correlation function of the currents corresponding to it,

$$\langle j_{\mathbf{k},\delta} j_{-\mathbf{k},\delta'}\rangle = T(\rho - \rho_s)(\delta\delta') + T\rho_s(\mathbf{k}\delta)(\mathbf{k}\delta')/k^2 \tag{25}$$

($j_{\mathbf{k},\delta}$ denotes the Fourier transform of the current $j_{\mathbf{r},\delta} = -\partial H/\partial v_{\mathbf{r},\delta}$; $\rho = \langle \partial^2 H/(\partial v_{\mathbf{r},\delta})^2\rangle$), express the fact that, as has been noted by many authors (see[5,10,11]), a "rigidity" with respect to transverse influences is the characteristic property of the superfluid state. Another characteristic property consists in the presence of spatially-inhomogeneous metastable states, described by an additional variable $\mathbf{v}_s(\mathbf{r})$ ("states associated with the superfluid current")[6]. Within the framework of the approximation which only takes the class $\mathscr{D}_0$ into consideration, these states are obtained from the states (7) by making the transformation $\varphi_{\mathbf{r}} \to \varphi_{\mathbf{r}} + \overline{\varphi}_{\mathbf{r}}$ (a displacement of the coordinates of the oscillators), where $\overline{\varphi}_{\mathbf{r}'} - \overline{\varphi}_{\mathbf{r}} = (\mathbf{v}_s(\mathbf{r}) \cdot \boldsymbol{\delta})$. The energy density ϵ and the entropy density s for the states with $\mathbf{v}_s \neq 0$ are given by

$$\varepsilon = \varepsilon_0 + {}^1/_2\rho_s v_s^2, \quad s(\varepsilon) = s_0(\varepsilon - {}^1/_2\rho_s v_s^2) \cong s_0 - {}^1/_2\rho_s v_s^2/T. \tag{26}$$

From (26) one can infer two-fluid hydrodynamics and the corresponding theory for magnetic substances (see[11,9]), also including Eq. (25), in exactly the same way as for three-dimensional systems. Thus, as illustrated by the two-dimensional systems under consideration, it is clear that, contrary to common opinion, a spontaneous violation of the phase symmetry is not necessary for the existence of superfluidity.[7]

2. ISOTROPIC SYSTEMS OF QUANTUM SPINS. HEISENBERG FERROMAGNETS

Let us show that the problem of calculating the long-range correlations for isotropic systems of quantum spins reduces to the corresponding problem for classical spins, which was solved in[1] (for low temperatures).

First let us consider a single spin. As is well known, the components $\hat{s}^{(\alpha)}$ (α = x, y, z) are the generators of the representations $\hat{\mathscr{D}}(u)$ of the group SU₂ ($\hat{\mathscr{D}}(u)$ is the matrix which corresponds in a given representation to the element u of the rotation group). One can parametrize the elements of the group SU₂, which is the universal covering group of the rotations, in the same way as the elements of the rotation group,—that is, by the Euler angles ϑ, φ, ψ, only the period with respect to the angle ψ will not be 2π, but 4π. If the irreducible representation corresponding to total spin S is considered, then the well-known orthogonality relations (see[14]) hold for the elements of the 2S + 1-dimensional matrices of the representation D(u); these relations will be used below in the following form (which is equivalent to the usual formulation): For any operator $\hat{F} = F_{MM'}$ acting on the 2S + 1-dimensional space of the representation, and for any normalized vector $|\cdot\rangle$ of this space, the following identity is satisfied:

$$\operatorname{Sp}\hat{F} = \sum_{M=-s}^{+s} F_{MM} = (2S+1)\int \langle\cdot|\hat{\mathscr{D}}^{+}(u)\hat{F}\hat{\mathscr{D}}(u)|\cdot\rangle(du), \tag{27}$$

where the integral is taken over the group SU₂, and $(du) = \sin\vartheta\, d\varphi d\psi/16\pi^2$ is the volume element on the group.

Now let us consider a lattice in which a spin $\hat{\mathbf{s}}_{\mathbf{r}}$ is associated with each lattice site **r** (all of the spins have the same magnitude, $(\hat{\mathbf{s}}_{\mathbf{r}})^2 = S(S+1)$). We associate its own independent SU₂-transformation $u_{\mathbf{r}}$ with each site **r**, and the corresponding operator $\hat{\mathscr{D}}_{\mathbf{r}}(u_{\mathbf{r}})$, which operates only on the variables pertaining to the spin $\hat{\mathbf{s}}_{\mathbf{r}}$. Then each lattice function $u = \{\vartheta, \varphi, \psi\}$ (i.e., the triplet of functions $\vartheta_{\mathbf{r}}$, $\varphi_{\mathbf{r}}$, $\psi_{\mathbf{r}}$) can be assigned an operator $\mathscr{D}\{U\} = \Pi\mathscr{D}_{\mathbf{r}}(u_{\mathbf{r}})$ (the product runs over all sites **r**) which transforms the states of the entire system. This transformation can be regarded as the same as a transformation of operators whereby each operator $\hat{F} = F(\ldots, \hat{\mathbf{s}}_{\mathbf{r}}, \ldots)$, which acts on the space of states of the system, corresponds to a transformed operator given by

$$\hat{F}\{u_{\cdot}\} = \hat{\mathscr{D}}^{+}\{u_{\cdot}\}\hat{F}\mathscr{D}\{u_{\cdot}\} = F(\ldots, \hat{s}_{\mathbf{r}}(u_{\mathbf{r}}), \ldots). \tag{28}$$

If the operator $\hat{F}$ is expressed in terms of $\hat{\mathbf{s}}_{\mathbf{r}}$, then the transformed operator (28) is obtained from this expression by replacing each of the $\hat{s}_{\mathbf{r}}^{(\alpha)}$ by

$$\hat{s}_{\mathbf{r}}^{(\alpha)}(u_{\mathbf{r}}) = \hat{\mathscr{D}}_{\mathbf{r}}^{+}(u_{\mathbf{r}})\hat{s}_{\mathbf{r}}^{(\alpha)}\hat{\mathscr{D}}_{\mathbf{r}}(u_{\mathbf{r}}) = R_{\alpha\beta}(u_{\mathbf{r}})\hat{s}_{\mathbf{r}}^{(\beta)}, \tag{29}$$

---

[6] According to Landau and Lifshitz (see [12], Sec. 81) a system has many phases if it has spatially-inhomogeneous equilibrium states. The critical point is best defined as the point at which a system of many phases becomes a single-phase system. This definition can be made completely rigorous by using the concept of "Gibbs states of infinite systems" which has been recently introduced (see [13]).

[7] One can describe the properties of the "two-dimensional crystals" considered in [1] in a similar fashion. The components of the deformation tensor are the additional thermodynamical variables which serve to describe the metastable states. Transverse waves exist in the system, so that this model describes a solid, notwithstanding the absence of long-range crystalline order.

where $R_{\alpha\beta}(u) = R_{\alpha\beta}(\vartheta, \varphi, \psi)$ is the matrix for a finite rotation characterized by the Euler angles ϑ, φ, and ψ (here and in what follows, summation over repeated Greek subscripts $\alpha, \beta, \ldots$ = x, y, z is to be understood).

We obtain the following result for the trace of an arbitrary operator $\hat{F}$, acting on the states of the entire system (by taking successively the traces with respect to the variables pertaining to each site, and each time applying Eq. (27)):

$$\operatorname{Sp}\hat{F} = (2S+1)^N \int\ldots\int \langle 0|\hat{F}\{u_{\cdot}\}|0\rangle \prod(du_{\mathbf{r}}). \tag{30}$$

Here N denotes the total number of lattice sites, $\hat{F}\{u\}$ is the operator given by Eq. (28), and $|0\rangle$ is any arbitrary normalized vector from the space of states of the lattice. Our discussion will be based on the utilization of the identity (30).

Let us consider a two-dimensional Heisenberg ferromagnet. Let

$$H = \frac{1}{2}J\sum\sum_{|\mathbf{r}-\mathbf{r}'|=a} \hat{s}_{\mathbf{r}}\hat{s}_{\mathbf{r}'} \tag{31}$$

denote the Hamiltonian of the system; we shall utilize the identity (30), having taken as $|0\rangle$ the ground state of (31) such that $\hat{z}_{\mathbf{r}}^{(z)}|0\rangle = +S|0\rangle$ for all **r**. For the average of an arbitrary operator $\hat{A}$ we then obtain

$$\langle\hat{A}\rangle = (2S+1)^N \int\ldots\int \left\langle 0 \left| \hat{A}\{u_{\cdot}\}\exp\left(\frac{\mathscr{F} - H\{u_{\cdot}\}}{T}\right)\right| 0\right\rangle \prod_{\mathbf{r}}(du_{\mathbf{r}}), \tag{32}$$

where $\mathscr{F}$ denotes the free energy, determined from the requirement $\langle 1\rangle = 1$. The integrand in Eq. (32) actually does not depend on the angles $\psi_{\mathbf{r}}$. In fact, from the identity

$$\hat{\mathscr{D}}(\vartheta, \varphi, \psi) = \mathscr{D}(\vartheta, \varphi, 0)\exp\{i\psi\hat{s}^{(z)}\}$$

it is clear that the states $\mathscr{D}\{u\}|0\rangle$ depend on $\psi_{\mathbf{r}}$ only through the factors $\exp\{i\psi_{\mathbf{r}}S\}$, which are cancelled in (32) by the conjugate factors coming from $\langle 0|\mathscr{D}^{+}\{u\}$. Therefore, in Eq. (32) one can omit the integrations over $d\psi_{\mathbf{r}}$ and by $u_{\mathbf{r}}$ understand $\{\vartheta_{\mathbf{r}}, \psi_{\mathbf{r}}, 0\}$, which simplifies the calculations.

Now let us introduce the functional $\Delta\mathscr{F}\{u\}$ defined by the equation

$$\left\langle 0\left|\exp\left(-\frac{1}{T}\hat{H}\{u_{\cdot}\}\right)\right|0\right\rangle = \exp\left(-\frac{E_0}{T}\right)\exp\left(-\frac{1}{T}\Delta\mathscr{F}\{u_{\cdot}\}\right) \tag{33}$$

(E_0 is the energy of the ground state). Furthermore, if we calculate the average of a product of local operators $\hat{A}_s$ pertaining to widely separated points $\mathbf{r}_s$ (such that $|\mathbf{r}_s - \mathbf{r}_{s'}| \gg a$) is calculated, then according to the principle of the decay of correlations (the ergodic property) we have in the present case (for the state $|0\rangle$)

$$\left\langle 0\left|\left(\prod_s \hat{A}_s(u_{\mathbf{r}_s})\right)\exp\left(-\frac{1}{T}\hat{H}\{u_{\cdot}\}\right)\right|0\right\rangle$$

$$\cong \left(\prod_s \langle 0|\hat{A}_s(u_{\mathbf{r}_s})|0\rangle\right)\exp(-E_0/T)\exp\left(-\frac{1}{T}\Delta\mathscr{F}\{u_{\cdot}\}\right). \tag{34}$$

Substituting this expression into (32), we see that the quantum-mechanical average of the product of local operators $\hat{A}_s$, pertaining to widely separated points $\mathbf{r}_s$, coincides with the average over the equivalent classical system, in which the configurations are described by functions $u_{\mathbf{r}} = \{\vartheta_{\mathbf{r}}, \varphi_{\mathbf{r}}, 0\}$, the energy of the configurations is given by the functional $\Delta\mathscr{F}\{u\}$, and the correspondence between physical quantities is realized according to the following rule: The quantum-mechanical quantity $\hat{A}_s$ is associated with the classical quantity

$$\mathscr{A}_s(\vartheta_{\mathbf{r}_s}, \varphi_{\mathbf{r}_s}) = \langle 0|\hat{A}_s(\vartheta_{\mathbf{r}_s}, \varphi_{\mathbf{r}_s}, 0)|0\rangle.$$

In particluar, here the components $s_{\mathbf{r}}^{(\alpha)}$ of the spin are related to the classical quantities $Sn_{\mathbf{r}}^{(\alpha)}$, where $\mathbf{n}_{\mathbf{r}}$ is a unit vector having the direction $\vartheta_{\mathbf{r}}$, $\varphi_{\mathbf{r}} - (\pi/2)$ in the spherical coordinate system:

$$\hat{s}_{\mathbf{r}} \to S\mathbf{n}_{\mathbf{r}} = S\{\sin\vartheta_{\mathbf{r}}\sin\varphi_{\mathbf{r}};\ -\sin\vartheta_{\mathbf{r}}\cos\varphi_{\mathbf{r}};\ \cos\vartheta_{\mathbf{r}}\}. \tag{35}$$

In order to use the results of the investigation of the classical system carried out in Sec. 2 of article[1], it is only necessary to determine the expression for the equivalent energy $\Delta\mathscr{F}\{u\}$. In this connection it is very important that the entire investigation in[1] was only based on the expression for the energy of slowly varying configurations, that is, configurations in which the differences $d_\delta u_{\mathbf{r}} = u_{\mathbf{r}+\delta} - u_{\mathbf{r}}$ are sufficiently small for $|\boldsymbol{\delta}| = a$.

One can find the expression for the equivalent energy of slowly varying configurations correct to within terms of second order in $d_\delta u_{\mathbf{r}}$ by using the corresponding expansion of the transformed Hamiltonian:

$$\hat{H}\{u_{\cdot}\} = \hat{H} + \hat{H}_1\{u_{\cdot}, d_\delta u_{\cdot}\} + \hat{H}_2\{u_{\cdot}, d_\delta u_{\cdot}\} + o((d_\delta u_{\cdot})^2), \tag{36}$$

where $\hat{H}$ is the initial Hamiltonian (31), $\hat{H}_1\{\ldots\}$ and $\hat{H}_2\{\ldots\}$ are the terms of first and second order in $d_\delta u_{\mathbf{r}}$ and $o(\ldots)$ are the terms of higher orders. Actually $\hat{H}\{u\}$ is obtained from (31) by making the substitution $\mathbf{s}_{\mathbf{r}} \cdot \mathbf{s}_{\mathbf{r}'} \to \mathbf{s}_{\mathbf{r}}(u_{\mathbf{r}}) \cdot \mathbf{s}_{\mathbf{r}'}(u_{\mathbf{r}'})$ for each pair of neighboring sites **r**, $\mathbf{r}' = \mathbf{r} + \boldsymbol{\delta}$. The term of zero order in the expansion of $\mathbf{s}_{\mathbf{r}}(u_{\mathbf{r}})\mathbf{s}_{\mathbf{r}+\delta}(u_{\mathbf{r}+\delta})$ in powers of $d_\delta u_{\mathbf{r}}$ is then equal to $\mathbf{s}_{\mathbf{r}}\mathbf{s}_{\mathbf{r}+\delta}$, because the scalar product does not change upon an identical rotation of both spins. Thus, the term of zero order in (36) coincides with the initial Hamiltonian (31) (which is simply a consequence of the invariance of H with respect to the simultaneous rotation of all the spins through identical angles). One can obtain explicit expressions for the terms of first and second order with respect to $d_\delta u_{\mathbf{r}}$ in Eq. (36) by using formula (29) and by calculating the expansion of $R_{\alpha\gamma}(u)R_{\beta\gamma}(u + du)$ in powers of du based on the well-known expressions for $R_{\alpha\beta}(u)$ in terms of the Euler angles (see[14]). We shall not present these calculations here, and we shall not even write down the final expressions for $H_1\{\ldots\}$ and $H_2\{\ldots\}$, limiting our attention to only a description of the method used to obtain these results.

In virtue of (36) one can use perturbation theory to calculate the values of the functional $\Delta\mathscr{F}\{u\}$ for slowly varying configurations, regarding the first term on the right hand side of (36) as the major term and treating the terms H_1 and H_2 as perturbations. Thus, we arrive at an expansion of $\Delta\mathscr{F}\{u\}$ in powers of the differences $d_\delta u_{\mathbf{r}}$, where the first-order term vanishes and the second-order term is given by

$$\Delta_2\mathscr{F}\{u_{\cdot}\} = \langle 0|H_2|0\rangle - \frac{T}{2}\int_0^{1/T} d\tau' \int_0^{\tau'} d\tau \langle 0|H_1 e^{-\tau(H-E_0)}H_1|0\rangle. \tag{37}$$

The second term on the right-hand side of Eq. (37) can be represented in the form of a summation over intermediate states, where only states corresponding to single spin waves with quasimomenta **k** give any contribution. The first term can also be represented in the form of a sum over **k**. Comparing these sums and taking into consideration that, as follows from[1], only wavelengths in the region $\ln(|\mathbf{k}|a) \leq -O(1/T)$ are important, one can see that the second term from (37) is negligible in this region in comparison with the first term, so that $\Delta_2\mathscr{F}\{u.\}$ is given simply by the average value $\langle 0|H_2\{u., d_\delta u.\}|0\rangle$. Evaluating this average value with the aid of the explicit expression for $H_2\{u., d_\delta u.\}$ (which is not written down above), and replacing $d_\delta\vartheta_r$ and $d_\delta\varphi_r$ by $(\delta\cdot\nabla)\vartheta_r$ and $(\delta\cdot\nabla)\varphi_r$, and also replacing the sum over r by an integral, we obtain the following result for the energy of the slowly changing configurations

$$\Delta_2\mathscr{F}\{u.\} \simeq {}^1/{}_2 JS^2 \int (dr)\,\{(\nabla\vartheta_r)^2 + (\nabla\varphi_r)^2 \sin^2\vartheta_r\}. \tag{38}$$

This expression coincides with the corresponding expression for the energy of the slowly varying classical configurations (see[1], formula (38)), on which the conclusions reached in[1] were based. Therefore, one can immediately apply these conclusions to the present case. In particular, for the asymptotic form of the pair correlation function we obtain (taking (35) into consideration):

$$\langle s_r s_{r'}\rangle \cong S^2 |(\mathbf{r}-\mathbf{r}')/r_0|^{-\alpha} \qquad (|\mathbf{r}-\mathbf{r}'| \gg a), \tag{39}$$

and for the specific magnetic moment in a weak external field, $h \ll J$, we obtain

$$\langle m\rangle = \langle S^{(z)}\rangle = S(ChS/J)^{\alpha/(4-\alpha)} + \ldots, \tag{40}$$

where $C \sim a^2/r_0^2$ is a certain constant, and in the present case $\alpha = T/\pi J$. The magnitude of the cutoff parameter r_0 and the constant C of Eq. (40) can be found from the correspondence with the expressions given by the theory of spin waves (the regions of applicability of Eqs. (39) and (40) and the theory of spin waves overlap for distances $\alpha \ln(|\mathbf{r}-\mathbf{r}'|/r_0) \ll 1$ and for fields $h \gtrsim T$). In this connection, for the cutoff parameter r_0 we obtain $r_0/a = (1/2)\sqrt{JS/T}$.

Other isotropic spin systems can be treated in analogous fashion, but since in the general case the exact ground state and the low-lying excited states are unknown, the constant appearing in front of (38) cannot be explicitly expressed in terms of the parameters of the initial Hamiltonian, and must be treated as a phenomenological parameter.

In conclusion I wish to thank A. I. Larkin, V. L. Pokrovskiĭ, and Yu. N. Ovchinnikov for a discussion of the questions considered in Sec. 1. V. N. Popov has courteously informed me that he has arrived at similar results (in regard to the superfluidity of a two-dimensional Bose liquid and the power-law nature of the asymptotic behavior), by starting from the formalism[15] developed by him.

---

[1] V. L. Berezinskiĭ, Zh. Eksp. Teor. Fiz. 59, 907 (1970) [Sov. Phys.-JETP 32, 493 (1971)].

[2] H. A. Bethe and E. Salpeter, Quantum Mechanics of One- and Two-electron Atoms, Springer-Verlag, 1957, Sec. 25 γ (Russian Transl., Fizmatgiz, 1960, p. 199).

[3] W. F. Vinen, in Quantum Fluids, edited by D. F. Brewer, North-Holland Publishing Co., 1966, pp. 74–108.

[4] Allen Miller, David Pines, and Philippe Nozières, Phys. Rev. **127**, 1452 (1962).

[5] David Pines, in Proceedings of the IXth International Conference on Low Temperature Physics, edited by J. G. Daunt, D. O. Edwards, F. J. Milford, and M. Yaqub, Plenum Press, 1965, Part A, pp. 61–78. (Russ. Transl., Nauka, 1967, p. 79).

[6] N. N. Bogolyubov, Supplement to Physica **26**, 1 (1960). N. N. Bogolyubov, Kvazisrednie v zadachakh statisticheskoĭ mekhaniki (Quasi-averages in Problems of Statistical Mechanics), Joint Institute for Nuclear Research, Dubna, R-511, 1963 (2nd ed.).

[7] J. W. Kane and L. P. Kadanoff, Phys. Rev. **155**, 80 (1967).

[8] Gordon Lasher, Phys. Rev. **172**, 224 (1968).

[9] B. I. Halperin and P. C. Hohenberg, Phys. Rev. **188**, 898 (1969).

[10] John Bardeen, in Handbuch der Physik, edited by S. Flügge, Vol. 15, Springer-Verlag, 1956, p. 274 (Russ. Transl., IIL, 1959). John Bardeen, Phys. Rev. Letters **1**, 399 (1958).

[11] P. C. Hohenberg and P. C. Martin, Ann. Phys. (N.Y.) **34**, 291 (1965).

[12] L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika (Statistical Physics), Nauka, 1964 (English Transl., Addison-Wesley, 2nd ed., 1969).

[13] R. L. Dobrushin, R. A. Minlos, and Yu. M. Sukhov, Appendix to the Russian translation of the book by David Ruelle: Statistical Mechanics, Rigorous Results, Mir, 1971.

[14] N. Ya. Vilenkin, Spetsial'nye funktsii i teoriya predstavleniĭ grupp (Special Functions and the Theory of Group Representations), Nauka, 1965, Chapter III, Sections 1, 2, 6 (English Transl., Vol. 22 of Translations of Mathematical Monographs, American Mathematical Society, 1968).

[15] V. N. Popov, ZhTMF 6, 90 (1970).

Translated by H. H. Nickle
117

SOVIET PHYSICS JETP VOLUME 34, NUMBER 3 MARCH, 1972

ELECTROACOUSTOMAGNETIC EFFECT AND HALL EFFECT IN SEMICONDUCTORS IN A STRONG ELECTRIC FIELD

Yu. P. MUKHORTOV and V. I. PUSTOVOĬT

All-union Institute of Physico-technical and Radiotechnical Measurements

Submitted February 22, 1971

Zh. Eksp. Teor. Fiz. 61, 1157–1167 (September, 1971)

Large-scale properties of a semiconducting crystal in external electric and magnetic fields are considered for the case when the carrier drift velocity exceeds the phase velocity of the sound wave and phonon generation occurs. It is shown that the generated phonon flux results in an additional force (besides the Lorentz force) which acts on the electrons and which is, generally speaking, not a potential force. It is precisely the nonpotentiality of the acoustoelectric force connected with the spatial anisotropy of the phonon emission diagram which leads to the appearance in the sample of an annular current component, and, as a consequence, to formation of a magnetic moment (electroacoustomagnetic effect). The Hall effect in a strong electric field under phonon-generation conditions is considered for two limiting cases: for Hall-shorted and Hall-open samples. It is shown that in the case of a Hall-open sample the Hall constant decreases and reverses sign with increasing electric field (the absolute value of the Hall constant in this case may exceed the value in a weak field). The current-voltage characteristic is also found, and it is shown that under phonon-generation conditions the current is saturated in a Hall-open sample. The case of a Hall-shorted sample with a Corbino disc geometry is considered and it is shown that in a strong magnetic field the current in the source circuit rises sharply. In weak magnetic fields the current is saturated. The theoretical results are compared with available experimental data and good qualitative agreement is found. Kinematic effects connected with phonon generation are also mentioned.

EXPERIMENTAL investigations of the Hall effect in semiconductors and semimetals in a strong electric field have shown that it is accompanied by a number of new singularities and phenomena which are quite difficult to interpret theoretically (see, for example, the monograph [1]). Most theoretical work on the Hall effect in a strong electric field initially begins with the premise that in a strong electric field the electron (hole) distribution function is significantly altered, so that the electric field can no longer be regarded as a small correction, and the electron temperature comes to depend on the electric field. By considering further some particular type of carrier scattering, with different dependences of the scattering time on the carrier energy, it is naturally possible to obtain different corrections to the Hall constant, necessitated by the action of the strong electric field. The galvanomagnetic properties and current-voltage characteristics of semiconductors in strong magnetic and electric fields were investigated in detail by Bass[2] (see also the references therein), who has shown that in a strong electric field, when the heating of the electrons is appreciable, the current-voltage characteristic of a semiconductor varies and it becomes possible to obtain negative differential resistance. Analogous phenomena were investigated earlier in plasma physics in connection with the problem of runaway electrons.[3]

At the same time, there is one more mechanism whereby the Hall constant can be altered in a strong electric field; this mechanism is connected with generation of acoustic phonons by supersonic drift of the electrons.[4] Indeed, if a sufficiently strong electric field is applied to the sample, such that the average directional velocity of the electrons or holes exceeds the phase velocity of the acoustic wave, then, as is well known,[5] acoustic phonons, sometimes also called acoustic noise,[1)] is produced in a crystal with relatively strong electron-phonon interaction. Owing to the acoustoelectric effect, the growing flux of acoustic phonons in the crystal reacts on the electrons, and an additional force produced by the generated phonons will act on the electrons in addition to the Lorentz force.[2)] Under these conditions the Hall emf and the Hall constant determined from it are significantly altered. Moreover, as will be shown below, a case is possible when the sign of the Hall constant is reversed (without a change in the type of carriers). Since the acoustoelectric force exerted on the electrons by the phonons generated in the crystal depends on the coordinates, it is natural that the experimentally measured Hall potential difference and Hall constant will also depend on the coordinates.

It should be noted that an investigation of the Hall effect in piezosemiconducting p-Te recently carried out by Tanaka and Hojo[8] has shown that in a strong electric field the Hall constant actually decreases and reverses sign, and that the region in which the indicated phenomena are observed corresponds quite closely to the region of phonon generation. Apparently, in a number of other experiments, where a decrease of the Hall constant

---

[1)] The influence of uneven distribution of the phonons in strong electric and magnetic fields was also investigated by Chuenkov [6], who has shown that the mutual dragging of the electrons and phonons has a strong influence on the galvanomagnetic properties of a semiconductor. However, he considered only the case of subsonic motion of the electrons, and there was no phonon generation.

[2)] The influence of an external monochromatic acoustic wave on different kinetic coefficients (particularly the Hall effect in an acoustoelectric current) was considered by Gulyaev [7].