Fluctuation kinetics in pure superconductors

A. G. Aronov and R. Katilyus

Leningrad Nuclear Physics Institute, USSR Academy of Sciences
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We construct a theory for the fluctuation kinetics in pure superconductors below the transition point. We assume that the frequency and wave vector satisfy the conditions $\hbar\omega \ll \Delta$, $q \ll \Delta/\hbar v_F$. The cause of the fluctuations is the collisions of the normal excitations with impurities and phonons. We study the spectrum and the spatial dispersion of the fluctuations in the absolute magnitude of the order parameter, in its phase, and in the superfluid velocity. We formulate fluctuation-dissipation theorems which connect the spectral densities of the fluctuations in these quantities with the corresponding impedance. We discuss the fluctuations in the magnetic flux in a superconducting ring and the effect of fluctuations in phase on the line width of the Josephson effect.

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1. INTRODUCTION

The present paper is devoted to a study of the fluctuation kinetics in pure superconductors. Practically the whole theory of fluctuations near equilibrium in normal metals reduces to giving the correlators of the random currents. As there are two new variables in superconductors which characterize the state of the superconductor, the phase χ and the absolute magnitude Δ of the order parameter, the theory is more complicated. All classical fluctuations in superconductors (and those are just the ones we shall consider in the present paper) are connected with the fluctuating motion of the normal excitations, on which the random potentials $\Phi = \frac{1}{2}\partial\chi/\partial t + e\varphi$ (φ is the electrostatic potential), $p_s = \frac{1}{2}(\nabla\chi - 2eA/c)$ and Δ act; these potentials themselves depend on the distribution function of the normal excitations.

We use the Langevin method for such a system, find general expressions for the equal time correlators, and formulate for a superconductor fluctuation-dissipation theorems which connect equal time correlators with the corresponding response functions. Moreover, we study the frequency dependence of the fluctuations in the whole temperature range up to the transition point and discuss the effect of phase fluctuations on the line width of Josephson radiation and the form of the current-voltage characteristics of a Josephson transition. Furthermore, we discuss the problem of magnetic flux fluctuations in superconducting rings.

There are at the moment a large number of papers, starting with the one by Aslamazov and Larkin[1] which are devoted to the effect of fluctuations near, and usually above, the transition point on the electrical properties of superconductors. Larkin and Ovchinnikov[2] have studied the order parameter fluctuation spectrum below the transition point when $\Delta \ll T$ in dirty superconductors. However, as far as we know, the fluctuation kinetics in the whole temperature range below the transition point in pure superconductors has not been considered.

2. LANGEVIN'S METHOD FOR DESCRIBING FLUCTUATIONS IN A SUPERCONDUCTOR

We use Langevin's method to describe fluctuations in a superconductor. If the wavevectors and frequencies of the excitations satisfy the conditions[1)] $\omega \ll 1$ and $q \ll \Delta/v_F$, the complete set of equations describing the behavior of the superconductor consists, as was shown earlier,[3] of

a) the kinetic equation for the distribution function of the excitations:

$$\frac{\partial n_{\mathbf{p}}}{\partial t}+\frac{\partial \tilde{\varepsilon}_{\mathbf{p}}}{\partial \mathbf{p}}\frac{\partial n_{\mathbf{p}}}{\partial \mathbf{r}}-\frac{\partial \tilde{\varepsilon}_{\mathbf{p}}}{\partial \mathbf{r}}\frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}}+I\{n_{\mathbf{p}}\}=0, \tag{2.1}$$

where

$$\tilde{\varepsilon}_{\mathbf{p}}=\varepsilon_{\mathbf{p}}+\mathbf{p}_s\mathbf{v}, \quad \varepsilon_{\mathbf{p}}=(\tilde{\xi}_{\mathbf{p}}^2+\Delta^2)^{1/2},$$
$$\tilde{\xi}_{\mathbf{p}}=\xi_{\mathbf{p}}+\Phi+p_s^2/2m, \quad \xi_{\mathbf{p}}=p^2/2m-\mu, \tag{2.2}$$

p is the quasi-momentum of the excitations, $I\{n_p\}$ is the operator for collisions of the excitations with phonons and impurities:

$$I\{n_{\mathbf{p}}\}=-2\pi\int\frac{d^3q}{(2\pi)^3}|C_{\mathbf{q}}|^2\{(1-n_{\mathbf{p}})n_{\mathbf{p}-\mathbf{q}}(u_{\mathbf{p}}u_{\mathbf{p}-\mathbf{q}}-v_{\mathbf{p}}v_{\mathbf{p}-\mathbf{q}})^2\cdot$$
$$\cdot[(1+N_{-\mathbf{q}})\delta(\tilde{\varepsilon}_{\mathbf{p}}-\tilde{\varepsilon}_{\mathbf{p}-\mathbf{q}}+\omega_{\mathbf{q}})+N_{\mathbf{q}}\delta(\tilde{\varepsilon}_{\mathbf{p}}-\tilde{\varepsilon}_{\mathbf{p}-\mathbf{q}}-\omega_{\mathbf{q}})]$$
$$-n_{\mathbf{p}}(1-n_{\mathbf{p}-\mathbf{q}})(u_{\mathbf{p}}u_{\mathbf{p}-\mathbf{q}}-v_{\mathbf{p}}v_{\mathbf{p}-\mathbf{q}})^2[N_{-\mathbf{q}}\delta(\tilde{\varepsilon}_{\mathbf{p}}-\tilde{\varepsilon}_{\mathbf{p}-\mathbf{q}}+\omega_{\mathbf{q}})$$
$$+(1+N_{\mathbf{q}})\delta(\tilde{\varepsilon}_{\mathbf{p}}-\tilde{\varepsilon}_{\mathbf{p}-\mathbf{q}}-\omega_{\mathbf{q}})]+(u_{\mathbf{p}}v_{\mathbf{p}-\mathbf{q}}+u_{\mathbf{p}-\mathbf{q}}v_{\mathbf{p}})^2\delta(\tilde{\varepsilon}_{\mathbf{p}}+\tilde{\varepsilon}_{-\mathbf{p}+\mathbf{q}}-\omega_{\mathbf{q}})\cdot$$
$$\cdot[(1-n_{\mathbf{p}})(1-n_{-\mathbf{p}+\mathbf{q}})N_{\mathbf{q}}-n_{\mathbf{p}}n_{-\mathbf{p}+\mathbf{q}}(1+N_{\mathbf{q}})]\}$$
$$-2\pi N_i\frac{(2\pi)^4}{m^2}\int\frac{d^3q}{(2\pi)^3}|\Lambda_{\mathbf{q}}|^2(n_{\mathbf{p}-\mathbf{q}}-n_{\mathbf{p}})(u_{\mathbf{p}}u_{\mathbf{p}-\mathbf{q}}-v_{\mathbf{p}}v_{\mathbf{p}-\mathbf{q}})^2\delta(\tilde{\varepsilon}_{\mathbf{p}}-\tilde{\varepsilon}_{\mathbf{p}-\mathbf{q}}), \tag{2.3}$$

N_i is the concentration of impurity atoms, C_q the matrix element of the electron-phonon interaction, Λ_q the amplitude for the scattering of an electron by an impurity atom in the normal metal,

$$u_{\mathbf{p}}^2=\tfrac{1}{2}(1+\tilde{\xi}_{\mathbf{p}}/\varepsilon_{\mathbf{p}}), \quad v_{\mathbf{p}}^2=\tfrac{1}{2}(1-\tilde{\xi}_{\mathbf{p}}/\varepsilon_{\mathbf{p}}); \tag{2.4}$$

b) the equation for the absolute magnitude of the order parameter:

$$1=-\frac{\lambda}{2}\int\frac{d^3p}{(2\pi)^3}\frac{1-2n_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}, \tag{2.5}$$

where $\lambda < 0$ is the effective electron-electron repulsion constant;

c) the continuity equation or the electro-neutrality equation, which is equivalent to it for a superconductor

$$\delta N=\delta\left\{\int d\tau_{\mathbf{p}}[u_{\mathbf{p}}^2n_{\mathbf{p}}+v_{\mathbf{p}}^2(1-n_{\mathbf{p}})]\right\}=0, \tag{2.6}$$

where $d\tau_p = 2d^3p/(2\pi)^3$. Together with the Maxwell equations, Eqs. (2.1), (2.5), and (2.6) form the complete set of equations.

When the superconductor is in a state of thermodynamic equilibrium, we can write the fluctuating correction to the distribution function of the normal excitations in the form

$$\delta n_{\mathbf{p}}=\delta\tilde{\varepsilon}_{\mathbf{p}}\frac{\partial n_0}{\partial \varepsilon_{\mathbf{p}}}+f_{\mathbf{p}}. \tag{2.7}$$

This correction leads to the appearance of the fluctuation potentials Φ, p_s, and $\delta\Delta$. Here

$$\delta\tilde{\varepsilon}_{\mathbf{p}}=\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\Phi+\mathbf{p}_s\mathbf{v}+\frac{\Delta}{\varepsilon_{\mathbf{p}}}\delta\Delta. \tag{2.8}$$

Linearizing (2.5) and (2.6) in terms of δn_p and the potentials we find that

$$\Phi=\frac{\partial\mu}{\partial N}\int d\tau_{\mathbf{p}}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}f_{\mathbf{p}}, \tag{2.9}$$

$$\delta\Delta=-\frac{\partial\mu}{\partial N}\frac{N}{N_s}\int d\tau_{\mathbf{p}}\frac{\Delta}{\varepsilon_{\mathbf{p}}}f_{\mathbf{p}}, \tag{2.10}$$

where

$$\frac{N_s}{N}=1+2\int_0^\infty d\xi_{\mathbf{p}}\frac{\partial n_0}{\partial \varepsilon_{\mathbf{p}}}$$

is the fraction of superconducting electrons.

To find the connection between p_s and f_p we use the Maxwell equation

$$\mathrm{rot}\,\mathbf{H}=4\pi\mathbf{J}/c \tag{2.11}$$

and the expression for the current

$$\mathbf{J}=e\mathbf{v}_sN_s+e\int d\tau_{\mathbf{p}}\,\mathbf{v}f_{\mathbf{p}}. \tag{2.12}$$

Changing to the Fourier components as far as r is concerned and using the fact that $H = -c\ \mathrm{curl}\ p_s/e$, we find that

$$\left[\delta_{ij}\left(q^2+\frac{1}{\lambda_L^2}\right)-q_iq_j\right]p_{sj}=-\frac{m}{\lambda_L^2N_s}\int d\tau_{\mathbf{p}}\,v_if_{\mathbf{p}}^{\mathbf{q}}(t), \tag{2.13}$$

where $\lambda_L^{-2} = 4\pi e^2N_S/mc^2$ is the square of the reciprocal of the London penetration depth. Or, introducing the matrix

$$K_{ij}(\mathbf{q})=q_iq_j-\delta_{ij}(q^2+1/\lambda_L^2),$$

we have

$$p_{si}=\frac{m}{\lambda_L^2N_s}K_{ij}^{-1}(\mathbf{q})\int d\tau_{\mathbf{p}}\,v_jf_{\mathbf{p}}^{\mathbf{q}}(t). \tag{2.14}$$

To obtain an equation for $f_p^q(t)$ we linearize the kinetic equation. It is then important that the collision operator vanishes if we substitute the equilibrium distribution function $n_0(\tilde{\epsilon}_p)$. The linearized kinetic equation then takes the form

$$\left(\frac{\partial}{\partial t}+i\mathbf{qv}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}+J_{\mathbf{p}}\right)f_{\mathbf{p}}^{\mathbf{q}}(t)+\frac{\partial\delta\tilde{\varepsilon}_{\mathbf{p}}}{\partial t}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}=G_{\mathbf{p}}^{\mathbf{q}}(t), \tag{2.15}$$

where J_p is the collision operator (2.3)—linearized with respect to the distribution function—taken in the zeroth approximation with respect to the fluctuating potentials. We have added in (2.15) the random force $G_p^q(t)$ in accordance with the Langevin method.

Changing to the Fourier components with respect to the time, we can write Eq. (2.15) in the form

$$\left\{-i\omega(1+\hat{L}_{\mathbf{p}})+i\mathbf{qv}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}+J_{\mathbf{p}}\right\}f_{\mathbf{p}}^{\omega\mathbf{q}}=G_{\mathbf{p}}^{\omega\mathbf{q}}, \tag{2.16}$$

where we have introduced the integral operator

$$\hat{L}_{\mathbf{p}}f_{\mathbf{p}}^{\omega\mathbf{q}}=\delta\tilde{\varepsilon}_{\mathbf{p}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}=\left[\frac{\partial\mu}{\partial N}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}\int d\tau_{\mathbf{p}_1}\frac{\xi_{\mathbf{p}_1}}{\varepsilon_{\mathbf{p}_1}}-\frac{\partial\mu}{\partial N}\frac{N}{N_s}\frac{\Delta}{\varepsilon_{\mathbf{p}}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}\int d\tau_{\mathbf{p}_1}\frac{\Delta}{\varepsilon_{\mathbf{p}_1}}\right.$$
$$\left.+\frac{m}{\lambda_L^2N_s}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}v_iK_{ij}^{-1}(\mathbf{q})\int d\tau_{\mathbf{p}_1}v_{1j}\right]f_{\mathbf{p}_1}^{\omega\mathbf{q}}. \tag{2.17}$$

To complete the construction of the Langevin scheme it is necessary to determine the correlators $\langle G_p(r, t)G_{p_1}(r_1, t_1)\rangle_{\omega q}$ of the random forces in the kinetic Eq. (2.16). To find the correlator of the random forces in the kinetic equation we use Onsager's method.[4] The probability for thermodynamic fluctuations is determined in our case by the change in the free energy of the superconductor and in the electromagnetic field when there are fluctuations. The change in the free energy when the distribution function fluctuates depends then on the fluctuation potentials which, according to (2.9), (2.10), and (2.14), are functionals of the fluctuation correction to the distribution function. The probability for fluctuations is thus given by varying the quantity[3]

$$\delta\mathcal{F}=\delta\left\{\int d\tau_{\mathbf{p}}\,\tilde{\varepsilon}_{\mathbf{p}}(n_{\mathbf{p}}-v_{\mathbf{p}}^2)-\frac{\Delta^2}{\lambda}-N\Phi+\frac{H^2}{8\pi}-TS\right\},$$
$$S=-\int d\tau_{\mathbf{p}}[n_{\mathbf{p}}\ln n_{\mathbf{p}}+(1-n_{\mathbf{p}})\ln(1-n_{\mathbf{p}})], \tag{2.18}$$

where S is the entropy of the gas of excitations.

Expanding $\delta\mathcal{F}$ up to and including second order terms and using the neutrality condition and the gap equation we get

$$\delta\mathcal{F}=-\frac{1}{2}\int d\tau_{\mathbf{p}}\frac{(f_{\mathbf{p}}^{\mathbf{q}})^2}{\partial n_0/\partial\varepsilon_{\mathbf{p}}}+\frac{N_s}{2N}\frac{\partial N}{\partial\mu}(\delta\Delta)^2-\frac{1}{2}\frac{\partial N}{\partial\mu}\Phi^2+N_s\frac{p_s^2}{2m}+\frac{H^2}{8\pi}$$
$$=-\frac{1}{2}\int\frac{d\tau_{\mathbf{p}}}{\partial n_0/\partial\varepsilon_{\mathbf{p}}}f_{\mathbf{p}}^{\mathbf{q}}(1+\hat{L}_{\mathbf{p}})f_{\mathbf{p}}^{\mathbf{q}}. \tag{2.19}$$

Now we can use the automatic scheme[4] for finding the random forces.

We introduce the generalized coordinates $x_p \equiv f_p^q$ and the generalized forces,

$$X_{\mathbf{p}}=\frac{1}{T}\frac{\partial\delta\mathcal{F}}{\partial x_{\mathbf{p}}}. \tag{2.20}$$

corresponding to them. Varying (2.19) and using the expressions for $\delta\Delta$, Φ, and p_s, we get

$$X_{\mathbf{p}}=-\frac{1}{T}\left(\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}\right)^{-1}(1+\hat{L}_{\mathbf{p}})f_{\mathbf{p}}^{\mathbf{q}}. \tag{2.21}$$

We write the relation between x_p and X_p, i.e., the kinetic Eq. (2.16) in the form[2)]

$$-i\omega f_{\mathbf{p}}^{\omega\mathbf{q}}=-\sum_{\mathbf{p}_1}(1+\hat{L}_{\mathbf{p}})^{-1}B_{\mathbf{p}}(\mathbf{q})(1+\hat{L}_{\mathbf{p}})^{-1}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}\cdot$$
$$\cdot\delta_{\mathbf{pp}_1}\left(\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}_1}}\right)^{-1}(1+\hat{L}_{\mathbf{p}_1})f_{\mathbf{p}_1}^{\omega\mathbf{q}}+(1+L_{\mathbf{p}})^{-1}G_{\mathbf{p}}^{\omega\mathbf{q}},$$
$$B_{\mathbf{p}}(\mathbf{q})=i\mathbf{qv}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}+J_{\mathbf{p}}, \tag{2.22}$$

whence we have, according to[4]

$$(1+\hat{L}_{\mathbf{p}})^{-1}(1+\hat{L}_{\mathbf{p}_1})^{-1}\langle G_{\mathbf{p}}G_{\mathbf{p}_1}\rangle_{\omega\mathbf{q}}=-\frac{T}{2\pi}\left[(1+\hat{L}_{\mathbf{p}})^{-1}B_{\mathbf{p}}(\mathbf{q})(1+\hat{L}_{\mathbf{p}})^{-1}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}\delta_{\mathbf{pp}_1}\right.$$
$$\left.+(1+\hat{L}_{\mathbf{p}_1})^{-1}B_{\mathbf{p}_1}^*(\mathbf{q})(1+\hat{L}_{\mathbf{p}_1})^{-1}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}_1}}\delta_{\mathbf{p}_1\mathbf{p}}\right]. \tag{2.23}$$

Operating on the left with the operator $(1+\hat{L}_p)(1+\hat{L}_{p_1})$ and using the fact that

$$(1+\hat{L}_{\mathbf{p}_1})\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}\delta_{\mathbf{pp}_1}=(1+\hat{L}_{\mathbf{p}})\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}_1}}\delta_{\mathbf{p}_1\mathbf{p}}$$

(this equation follows directly from the definition (2.17)), we get

$$\langle G_{\mathbf{p}}G_{\mathbf{p}_1}\rangle_{\omega\mathbf{q}}=-\frac{T}{2\pi}(J_{\mathbf{p}}+J_{\mathbf{p}_1})\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}\delta_{\mathbf{pp}_1}. \tag{2.24}$$

It is necessary to note here the following important fact. If the potentials Φ, p_s, and $\delta\Delta$ were not self-consistent, their fluctuations, and also the fluctuations in the distribution function, would be statistically independent, and the square of the fluctuations in the distribution function would be given by the usual expression

$$\langle\delta n_{\mathbf{p}}\delta n_{\mathbf{p}_1}\rangle=-T\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}\delta_{\mathbf{pp}_1}.$$

In superconductors, however,

$$\langle\delta n_{\mathbf{p}}\delta n_{\mathbf{p}_1}\rangle=-T\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}(1+\hat{L}_{\mathbf{p}})\delta_{\mathbf{pp}_1}$$

or

$$\langle\delta n_{\mathbf{p}}\delta n_{\mathbf{p}_1}\rangle=-T\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}\delta_{\mathbf{p}\mathbf{p}_1}-T\frac{\partial\mu}{\partial N}\frac{\xi_{\mathbf{p}}\xi_{\mathbf{p}_1}}{\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{p}_1}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}_1}}$$
$$+T\frac{\partial\mu}{\partial N}\frac{N}{N_s}\frac{\Delta^2}{\varepsilon_{\mathbf{p}}\varepsilon_{\mathbf{p}_1}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}_1}}-T\frac{m}{\lambda_L{}^2N_s}v_iK_{ij}{}^{-1}v_{1j}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}_1}}. \qquad (2.25)$$

This is obvious, as the gas of the excitations is situated in self-consistent fields and interacts with them. In this sense, it is not a perfect gas of excitations. At the same time the correlator of the random forces is determined solely through the collision operator, as random motions occur only when we take collisions into account and over times of the order of the collision time, while the distribution function, and thus also the fluctuation potentials, change over times of the order of the time between collisions which in our approximation are much longer than the collision times.

3. CORRELATORS OF THE FLUCTUATIONS IN THE BASIC PARAMETERS OF A SUPERCONDUCTOR

The spectral density of the fluctuation correlator is

$$\langle f_{\mathbf{p}}f_{\mathbf{p}_1}\rangle_{\omega\mathbf{q}}=\int_0^\infty e^{i\omega\tau}\,d\tau\langle f_{\mathbf{p}}(\tau)f_{\mathbf{p}_1}(0)\rangle_{\mathbf{q}}+\int_0^\infty e^{-i\omega\tau}\langle f_{\mathbf{p}_1}(\tau)f_{\mathbf{p}}(0)\rangle_{\mathbf{q}}\,d\tau. \qquad (3.1)$$

From the kinetic Eq. (2.15) we have

$$f_{\mathbf{p}}{}^{\omega\mathbf{q}}=\frac{1}{-i\omega+B_{\mathbf{p}}(\mathbf{q})}\left\{G_{\mathbf{p}}{}^{\omega\mathbf{q}}+i\omega\left[\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\Phi^{\omega\mathbf{q}}+\mathbf{v}\mathbf{p}_s{}^{\omega\mathbf{q}}+\frac{\Delta}{\varepsilon_{\mathbf{p}}}\delta\Delta^{\omega\mathbf{q}}\right]\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}\right\}. \qquad (3.2)$$

Substituting (3.2) into (2.10) we find that

$$\delta\Delta^{\omega\mathbf{q}}=-\frac{1}{\gamma_{\mathbf{q}\omega}}\frac{\partial\mu}{\partial N}\frac{N}{N_s}\int d\tau_{\mathbf{p}}\frac{\Delta}{\varepsilon_{\mathbf{p}}}\frac{1}{-i\omega+B_{\mathbf{p}}(\mathbf{q})}G_{\mathbf{p}}{}^{\omega\mathbf{q}}. \qquad (3.3)$$

In obtaining (3.3) we used the fact that the operator $\hat{J}_{\mathbf{p}}$ conserves the parity of the function on which it operates, with respect to $\mathbf{p}$ and $\xi_{\mathbf{p}}$, as can be verified by a direct calculation. All integrals containing odd powers of $\mathbf{p}$ or $\xi_{\mathbf{p}}$ in the numerator are thus identically equal to zero. Here

$$\gamma_{\mathbf{q}\omega}=1+\frac{\partial\mu}{\partial N}\frac{N}{N_s}\int d\tau_{\mathbf{p}}\frac{\Delta}{\varepsilon_{\mathbf{p}}}\frac{i\omega}{-i\omega+B_{\mathbf{p}}(\mathbf{q})}\frac{\Delta}{\varepsilon_{\mathbf{p}}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}. \qquad (3.4)$$

Using (3.3), the definition (3.1), and Eq. (2.24) for the random force correlator we get

$$\langle\delta\Delta^2\rangle_{\omega\mathbf{q}}=\frac{T}{\pi\omega}\frac{\partial\mu}{\partial N}\frac{N}{N_s}\operatorname{Im}\frac{1}{\gamma_{\mathbf{q}\omega}}. \qquad (3.5)$$

In obtaining (3.5) we used the identity

$$\int d\tau_{\mathbf{p}}\,d\tau_{\mathbf{p}_1}A_{\mathbf{p}}C_{\mathbf{p}_1}\frac{1}{-i\omega+B_{\mathbf{p}}(\mathbf{q})}\frac{1}{i\omega+B_{\mathbf{p}_1}{}^*(\mathbf{q})}\langle G_{\mathbf{p}}G_{\mathbf{p}_1}\rangle_{\omega\mathbf{q}}$$
$$=-\frac{T}{2\pi}\int d\tau_{\mathbf{p}}\left[A_{\mathbf{p}}\frac{1}{-i\omega+B_{\mathbf{p}}(\mathbf{q})}C_{\mathbf{p}}+C_{\mathbf{p}}\frac{1}{i\omega+B_{\mathbf{p}}{}^*(\mathbf{q})}A_{\mathbf{p}}\right]\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}. \qquad (3.6)$$

Expression (3.5) is also the required expression for the spectrum of the fluctuations in the absolute magnitude of the order parameter. We give its analysis below.

Using (2.9) and (3.2) we find that

$$\Phi^{\omega\mathbf{q}}=\frac{\mathbf{w}\mathbf{p}_s{}^{\omega\mathbf{q}}}{a_{\mathbf{q}\omega}}+\frac{1}{a_{\mathbf{q}\omega}}\frac{\partial\mu}{\partial N}\int d\tau_{\mathbf{p}}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\frac{1}{-i\omega+B_{\mathbf{p}}(\mathbf{q})}G_{\mathbf{p}}{}^{\omega\mathbf{q}}, \qquad (3.7)$$

where

$$a_{\mathbf{q}\omega}=1-\frac{\partial\mu}{\partial N}\int d\tau_{\mathbf{p}}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\frac{i\omega}{-i\omega+B_{\mathbf{p}}(\mathbf{q})}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}, \qquad (3.8)$$

$$w_i=\frac{\partial\mu}{\partial N}\int d\tau_{\mathbf{p}}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\frac{i\omega}{-i\omega+B_{\mathbf{p}}(\mathbf{q})}v_i\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}. \qquad (3.9)$$

The vector $\mathbf{w}$ which has the dimensions of a velocity is, in an isotropic medium, directed along $\mathbf{q}$ and vanishes as $\mathbf{q}\to 0$.

The matrix elements of the linearized collision operator satisfy the equation[3]

$$J_{\mathbf{p}\mathbf{p}_1}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}_1}}=J_{-\mathbf{p}_1,-\mathbf{p}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}. \qquad (3.10)$$

Close to equilibrium it follows from the time reversal symmetry of the laws of mechanics and is verified in every actual case. From (3.10) follows the following equation for the operator $B_{\mathbf{p}}(\mathbf{q})$:

$$B_{\mathbf{p}\mathbf{p}_1}(\mathbf{q})\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}_1}}=B^*_{-\mathbf{p}_1,-\mathbf{p}}(\mathbf{q})\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}, \qquad (3.11)$$

whence we have for the matrix elements of the inverse operator $[-i\omega\hat{1}+\hat{B}(\mathbf{q})]^{-1}$ ($\hat{1}$ is the unit matrix)

$$\left(\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}\right)^{-1}[-i\omega\hat{1}+\hat{B}(\mathbf{q})]^{-1}_{\mathbf{p}\mathbf{p}_1}=\left(\frac{\partial n_0}{\partial\varepsilon}\right)^{-1}[-i\omega\hat{1}+\hat{B}^*(\mathbf{q})]^{-1}_{-\mathbf{p}_1,-\mathbf{p}}. \qquad (3.12)$$

Applying relation (3.12) to (3.9) we have

$$w_i=\frac{\partial\mu}{\partial N}\int d\tau_{\mathbf{p}}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\frac{i\omega}{-i\omega+B_{\mathbf{p}}(\mathbf{q})}v_i\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}$$
$$=-\frac{\partial\mu}{\partial N}\int d\tau_{\mathbf{p}}\,v_i\frac{i\omega}{-i\omega+B_{\mathbf{p}}{}^*(\mathbf{q})}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}=-\overline{w}_i{}^*(-\omega,\mathbf{q}). \qquad (3.13)$$

Using (2.14), (3.2), and (3.7) we get for $p_{si}^{\omega\mathbf{q}}$ the following equation:

$$K_{ij}(\omega,\mathbf{q})p_{sj}{}^{\omega\mathbf{q}}=\frac{4\pi e^2}{c^2}\int d\tau_{\mathbf{p}}\left(v_i-\frac{w_i}{a_{\mathbf{q}\omega}}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\right)\frac{1}{-i\omega+B_{\mathbf{p}}(\mathbf{q})}G_{\mathbf{p}}{}^{\omega\mathbf{q}}, \qquad (3.14)$$

whence, using (3.6), we have after simple transformations

$$\langle p_{si}p_{sj}\rangle_{\omega\mathbf{q}}=-\frac{T}{2\pi i\omega}\frac{4\pi e^2}{c^2}\{K_{ij}{}^{-1}(\omega,\mathbf{q})-K_{ji}^{-1*}(\omega,\mathbf{q})\}, \qquad (3.15)$$

$$K_{ij}(\omega,\mathbf{q})=q_iq_j-\delta_{ij}\left(q^2+\frac{1}{\lambda_L{}^2}\right)+\frac{4\pi i\omega}{c^2}\sigma_{ij}(\omega,\mathbf{q})-\frac{4\pi e^2}{c^2}\frac{\partial N}{\partial\mu}\frac{\overline{w}_iw_j}{a_{\mathbf{q}\omega}}, \qquad (3.16)$$

$$\sigma_{ij}(\omega,\mathbf{q})=-e^2\int d\tau_{\mathbf{p}}\,v_i\frac{1}{-i\omega+B_{\mathbf{p}}(\mathbf{q})}v_j\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}=\sigma_{ji}(\omega,-\mathbf{q}). \qquad (3.17)$$

The quantity $\sigma_{ij}(\omega,\mathbf{q})$ is the same as the contribution of the normal excitations in a superconductor to the high-frequency conductivity.[5]

Using (3.7) and (3.14) we find that

$$\langle\Phi^2\rangle_{\omega\mathbf{q}}=-\frac{T}{\pi\omega}\frac{\partial\mu}{\partial N}\operatorname{Im}\left[\frac{1}{a_{\mathbf{q}\omega}}\left(1+\frac{4\pi e^2}{c^2}\frac{\partial N}{\partial\mu}\frac{w_iK_{ij}{}^{-1}(\omega,\mathbf{q})\overline{w}_j}{a_{\mathbf{q}\omega}}\right)\right]. \qquad (3.18)$$

The electrical field E_i is related to p_{si} and Φ through the equation

$$eE_i=-i\omega p_{si}-iq_i\Phi, \qquad (3.19)$$

whence the correlator of the fluctuations in the transverse fields is

$$\langle E_\perp{}^2\rangle_{\omega\mathbf{q}}=-\frac{T}{\pi}\frac{4\pi\omega}{c^2}\operatorname{Im}K_\perp{}^{-1}(\omega,\mathbf{q})=\frac{T}{\pi}\operatorname{Re}Z_\perp(\omega,\mathbf{q}), \qquad (3.20)$$

and the correlator of the fluctuations in the longitudinal fields

$$\langle E_\parallel{}^2\rangle_{\omega\mathbf{q}}=-\frac{T}{\pi\omega}\operatorname{Im}\left\{\frac{q^2}{a_{\mathbf{q}\omega}}\frac{\partial\mu}{\partial N}+\frac{4\pi e^2}{c^2}K_\parallel{}^{-1}(\omega,\mathbf{q})\right. \qquad (3.21)$$
$$\left.\times\left[1+\frac{\omega q_i(w_i+\overline{w}_i)}{a_{\mathbf{q}\omega}}+\frac{q_iq_jw_i\overline{w}_j}{a_{\mathbf{q}\omega}{}^2}\right]\right\}=\frac{T}{\pi}\operatorname{Re}Z_\parallel(\omega,\mathbf{q}),$$

where $Z_\perp(\omega,\mathbf{q})$ and $Z_\parallel(\omega,\mathbf{q})$ are the impedances for transverse and longitudinal perturbations.

Equations (3.20) and (3.21) from the fluctuation-dissipation theorem for a superconductor which connects the field fluctuations with the impedance $Z(\omega,\mathbf{q})$. It is clear from (3.20) that it has in a superconductor the usual form[6], notwithstanding the infinite zero-frequency conductivity which is reflected in the occurrence of a term proportional[3)] to $c^2/4\pi i\omega\lambda_L^2$.

Equations (3.5), (3.15), (3.18), and (3.20) describe the spectrum of the fluctuations in a superconductor and they have the form of fluctuation-dissipation theorems, connecting the correlators of fluctuations with the imaginary parts of the corresponding susceptibilities. We emphasize here that if the spatial dispersion is

taken into account all expressions are obtained under the assumption that $q \ll \Delta/v_F$, the reciprocal coherence length. In type-I superconductors terms containing q^2 must thus be dropped, since $\lambda_L < v_F/\Delta \ll q^{-1}$.

4. FLUCTUATION SPECTRA

As we have already noted above the correlators of the fluctuations in the parameters of a superconductor can be expressed in terms of the imaginary parts of the appropriate susceptibilities. We can thus expect that for the quantities appearing on the right-hand sides of Eqs. (3.5), (3.15), and (3.18) the Kramers-Kronig relations hold. The derivation of the corresponding formulae does not differ from the standard one[4] if one notes that the spectral functions are, according to (3.1), defined as retarded functions, and therefore have no poles in the upper half-plane. However, as the susceptibilities have finite limits as $\omega\to\infty$, we must write the corresponding relations with the necessary subtractions. We find for instance, from the Kramers-Kronig relations, that

$$\frac{1}{2}[a^{-1}(0)-a^{-1}(\infty)]=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi i\omega}a_\omega{}^{-1}, \qquad (4.1)$$

where $a^{-1}(0)=a^{-1}_{\omega=0}$, $a^{-1}(\infty)=a^{-1}_{\omega=\infty}$.

Using relations such as (4.1) we get for the equal-time correlators the following expressions:

$$\langle\delta\Delta^2\rangle_{\mathbf{q}}=-T\frac{\partial\mu}{\partial N}\frac{N}{N_s}\left[1-\frac{\partial\mu}{\partial N}\frac{N}{N_s}\int d\tau_{\mathbf{p}}\frac{\Delta^2}{\varepsilon_{\mathbf{p}}{}^2}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}\right]^{-1}$$
$$\cdot\frac{\partial\mu}{\partial N}\frac{N}{N_s}\int d\tau_{\mathbf{p}}\frac{\Delta^2}{\varepsilon_{\mathbf{p}}{}^2}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}, \qquad (4.2)$$

$$\langle\Phi^2\rangle_{\omega\mathbf{q}}=-T\frac{\partial\mu}{\partial N}\left[1+\frac{\partial\mu}{\partial N}\int d\tau_{\mathbf{p}}\frac{\xi_{\mathbf{p}}{}^2}{\varepsilon_{\mathbf{p}}{}^2}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}\right]^{-1}\frac{\partial\mu}{\partial N}\int d\tau_{\mathbf{p}}\frac{\xi_{\mathbf{p}}{}^2}{\varepsilon_{\mathbf{p}}{}^2}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}, \qquad (4.3)$$

$$\langle p_{s\parallel}{}^2\rangle_{\mathbf{q}}=\frac{mT}{N_s}\frac{N_n}{N},\quad\langle p_{s\perp}{}^2\rangle_{\mathbf{q}}=\frac{mT}{N_s}\frac{N_n}{N}(1+q^2\lambda_L{}^2)^{-1}\left(1+q^2\lambda_L{}^2\frac{N_s}{N}\right)^{-1}. \qquad (4.4)$$

At low temperatures

$$\langle\Phi^2\rangle_{\mathbf{q}}=\frac{T^2}{\Delta}\frac{\partial\mu}{\partial N}\frac{N_n}{N},\quad\langle\delta\Delta^2\rangle_{\mathbf{q}}=T\frac{\partial\mu}{\partial N}\frac{N_n}{N}, \qquad (4.5)$$

where $N_n = N - N_s$ is the density of the normal component which at low temperatures is equal to $N_n/N = (2\pi\Delta/T)^{1/2}e^{-\Delta/T}$. At high temperatures, $\Delta \ll T$,

$$\langle\Phi^2\rangle_{\mathbf{q}}=\frac{4T^2}{\pi\Delta}\frac{\partial\mu}{\partial N},\quad\langle\delta\Delta^2\rangle_{\mathbf{q}}=T\frac{\partial\mu}{\partial N}\frac{N}{N_s}. \qquad (4.6)$$

Equations (4.6) are the same as the corresponding expressions obtained from the Ginzburg-Landau equation. At low temperatures the intensity of the fluctuations is proportional to the density of the normal component.

We turn now to a study of the frequency-dependence of the fluctuations. We start with the fluctuations in the absolute magnitude of the order parameter. When $q = 0$

$$\langle\delta\Delta^2\rangle_\omega\sim\operatorname{Im}\gamma_\omega{}^{-1},$$

$$\gamma_\omega=1+\frac{\partial\mu}{\partial N}\frac{N}{N_s}\int d\tau_{\mathbf{p}}\frac{\Delta}{\varepsilon_{\mathbf{p}}}\frac{i\omega}{-i\omega+J_{\mathbf{p}}}\frac{\Delta}{\varepsilon_{\mathbf{p}}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}. \qquad (4.7)$$

The linearized collision operator occurring in Eq. (4.7) acts on a function depending only on the energy $\epsilon_{\mathbf{p}}$ (even function of $\xi_{\mathbf{p}}$). This means that the whole of the frequency dispersion of the correlator $\langle\delta\Delta^2\rangle_\omega$ is connected with the inelastic scattering by phonons as the operator of elastic collisions only reduces any function depending on the energy to zero.

To find the frequency dependence of the fluctuation spectrum $\langle\delta\Delta^2\rangle_\omega$ it is necessary to solve the equation

$$\{-i\omega+J_{\mathbf{p}}\}\psi_{\mathbf{p}}=\frac{\Delta}{\varepsilon_{\mathbf{p}}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}. \qquad (4.8)$$

If $\psi_{\mathbf{p}} = \varphi^{(+)}(\epsilon_{\mathbf{p}})\partial n_0/\partial\epsilon_{\mathbf{p}}$ we get, using (2.3) and linearizing with respect to $\psi_{\mathbf{p}}$, after simple, but tedious transformations the following expression for the photon operator $J_{ph}\{\varphi^{(+)}(\epsilon_{\mathbf{p}})\}$, acting on an even function of $\xi_{\mathbf{p}}$:[7]

$$J_{\mathbf{p}}\varphi^{(+)}(\varepsilon_{\mathbf{p}})\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}\equiv J_{ph}\{\varphi^{(+)}(\varepsilon_{\mathbf{p}})\}=-\frac{n_0(\varepsilon)}{4ms^2p_Fl_aT^2}\left\{\int_\Delta^\varepsilon\frac{\varepsilon'\,d\varepsilon'}{(\varepsilon'^2-\Delta^2)^{1/2}}\left(1-\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\right.$$
$$\times(\varepsilon-\varepsilon')^2\left[1+\exp\left(-\frac{\varepsilon'}{T}\right)\right]^{-1}\left[\exp\left(\frac{\varepsilon'-\varepsilon}{T}\right)-1\right]^{-1}[\varphi^{(+)}(\varepsilon')-\varphi^{(+)}(\varepsilon)]$$
$$-\int_\varepsilon^\infty\frac{\varepsilon'\,d\varepsilon'}{(\varepsilon'^2-\Delta^2)^{1/2}}\left(1-\frac{\Delta^2}{\varepsilon\varepsilon'}\right)(\varepsilon-\varepsilon')^2\left[1+\exp\left(-\frac{\varepsilon'}{T}\right)\right]^{-1} \qquad (4.9)$$
$$\times\left[\exp\left(\frac{\varepsilon'-\varepsilon}{T}\right)-1\right]^{-1}[\varphi^{(+)}(\varepsilon')-\varphi^{(+)}(\varepsilon)]+\int_\Delta^\infty\frac{\varepsilon'\,d\varepsilon'}{(\varepsilon'^2-\Delta^2)^{1/2}}\left(1+\frac{\Delta^2}{\varepsilon\varepsilon'}\right)$$
$$\left.\times(\varepsilon+\varepsilon')^2\left[1-\exp\left(-\frac{\varepsilon'+\varepsilon}{T}\right)\right]^{-1}\left[\exp\left(\frac{\varepsilon'}{T}\right)+1\right]^{-1}[\varphi^{(+)}(\varepsilon')+\varphi^{(+)}(\varepsilon)]\right\}.$$

Here l_a is the mean free path of the conduction electrons in a normal conductor when $sp_F \ll T$, s is the sound velocity,

$$|c_{\mathbf{q}}|^2=\pi\omega_{\mathbf{q}}/2Tm^2l_a.$$

As in[7], we easily find a solution of Eq. (4.8) with the operator (4.9) for $\Delta \ll T$ in the energy range $\epsilon\sim\Delta$. We note that just that range is important for us for the study of the fluctuation spectrum as the integral occurring in γ_ω converges at energies $\epsilon\sim\Delta$ when $\Delta \ll T$. For the solution we use a method similar to one used earlier.[8]

In the energy range $\epsilon \ll T$ the first of the integrals in (4.9) is small compared to the other two and we drop it. In the other two integrals we neglect ϵ and Δ as compared to $\epsilon'\sim T$. With the same accuracy we replace the lower limits by zero. Altogether the terms taken into account cancel one another and we get the following expression:

$$J_{ph}\{\varphi^{(+)}(\varepsilon_{\mathbf{p}})\}=-\varphi^{(+)}(\varepsilon)n_0(\varepsilon)/2T\tau_0, \qquad (4.10)$$

where $\tau_0^{-1} = 7\,\zeta(3)T^2/4ms^2p_Fl_a$, $\zeta(x)$ is the Riemann zeta function.

The solution of Eq. (4.8) with the operator (4.10) is trivial in the region $\epsilon \ll T$. Using the solution of (4.8) and performing the integration in (4.7) we find from (3.5) that

$$\langle\delta\Delta^2\rangle_\omega=\frac{T}{\pi}\frac{\partial\mu}{\partial N}\frac{N}{N_s}\frac{\tau_\Delta}{1+\omega^2\tau_\Delta{}^2},\quad\tau_\Delta=\tau_0\frac{\pi\Delta}{4T}\frac{N}{N_s}. \qquad (4.11)$$

Thus, when we approach the transition point the width of the fluctuation spectrum decreases proportional to $(T_c - T)^{1/2}$; τ_Δ is the relaxation time of the absolute magnitude of the order parameter close to the transition point, found by Schmid.[9]

We now turn to a study of the frequency dependence of the fluctuation spectrum $\langle\Phi^2\rangle_\omega$:

$$\langle\Phi^2\rangle_\omega=-\frac{T}{\pi\omega}\frac{\partial\mu}{\partial N}\operatorname{Im}\frac{1}{a_\omega}, \qquad (4.12)$$

$$a_\omega=1-\frac{\partial\mu}{\partial N}\int d\tau_{\mathbf{p}}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\frac{i\omega}{-i\omega+J_{\mathbf{p}}}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}. \qquad (4.13)$$

First of all we draw attention to the fact that the collision operator occurring in the expression for $\langle\Phi^2\rangle_\omega$ acts upon an odd function of $\xi_{\mathbf{p}}$. This means that out of the frequency spectrum $\langle\Phi^2\rangle_\omega$, again responsible for the relaxation by phonons, we take that part which is odd in $\xi_{\mathbf{p}}$.

To find the fluctuation spectrum we must solve the equation

$$\{-i\omega+J_{\mathbf{p}}\}\chi_{\mathbf{p}}=\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}. \quad (4.14)$$

If $\chi_p = \operatorname{sign}\xi_p\varphi^{(-)}(\epsilon_p)\partial n_0/\partial\epsilon_p$, we get, using (2.3), the following expression for the linearized collision operator:

$$J_{ph}\{\varphi^{(-)}(\varepsilon_{\mathbf{p}})\}=\frac{\operatorname{sign}\xi_{\mathbf{p}}n_0(\varepsilon)}{4ms^2p_Fl_aT^2}\left\{\int_\Delta^\varepsilon\frac{\varepsilon'\,d\varepsilon'}{(\varepsilon'^2-\Delta^2)^{1/2}}(\varepsilon-\varepsilon')^2\left[1+\exp\left(-\frac{\varepsilon'}{T}\right)\right]^{-1}\right.$$

$$\times\left[\exp\left(\frac{\varepsilon'-\varepsilon}{T}\right)-1\right]^{-1}\left[\left(1-\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\varphi^{(-)}(\varepsilon)-\frac{(\varepsilon^2-\Delta^2)^{1/2}(\varepsilon'^2-\Delta^2)^{1/2}}{\varepsilon\varepsilon'}\varphi^{(-)}(\varepsilon')\right]$$

$$-\int_\varepsilon^\infty\frac{\varepsilon'\,d\varepsilon'}{(\varepsilon'^2-\Delta^2)^{1/2}}(\varepsilon-\varepsilon')^2\left[1+\exp\left(-\frac{\varepsilon'}{T}\right)\right]^{-1}\left[\exp\left(\frac{\varepsilon'-\varepsilon}{T}\right)-1\right]^{-1}$$

$$\times\left[\left(1-\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\varphi^{(-)}(\varepsilon)-\frac{(\varepsilon^2-\Delta^2)^{1/2}(\varepsilon'^2-\Delta^2)^{1/2}}{\varepsilon\varepsilon'}\varphi^{(-)}(\varepsilon')\right]-\int_\Delta^\infty\frac{\varepsilon'\,d\varepsilon'}{(\varepsilon'^2-\Delta^2)^{1/2}}$$

$$\times(\varepsilon+\varepsilon')^2\left[1-\exp\left(-\frac{\varepsilon+\varepsilon'}{T}\right)\right]^{-1}\left[\exp\left(\frac{\varepsilon'}{T}\right)+1\right]^{-1} \quad (4.15)$$

$$\left.\times\left[\left(1+\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\varphi^{(-)}(\varepsilon)-\frac{(\varepsilon^2-\Delta^2)^{1/2}(\varepsilon'^2-\Delta^2)^{1/2}}{\varepsilon\varepsilon'}\varphi^{(-)}(\varepsilon')\right]\right\}$$

We consider first of all the high-temperature case when $\Delta \ll T$. To do this we use the following method. We assume that the characteristic width of the fluctuation spectrum τ_Φ^{-1} is much larger than the characteristic scale length of the collision operator τ_S^{-1} which appears when it operates on the function $(\xi_p/\epsilon_p)\partial n_0/\partial\epsilon_p$. We can then expand (4.13) in terms of the small parameter $1/\omega\tau_S$ and write (4.12) in the form

$$\langle\Phi^2\rangle_\omega=\frac{T}{\pi}\frac{\partial\mu}{\partial N}\frac{4T}{\pi\Delta}\frac{\tau_\Phi}{1+\omega^2\tau_\Phi^2}, \quad (4.16)$$

$$\tau_\Phi^{-1}=-\frac{4T}{\pi\Delta}\frac{\partial\mu}{\partial N}\int d\tau_{\mathbf{p}}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}J_{\mathbf{p}}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}. \quad (4.17)$$

It is clear from (4.17) that, indeed, $\tau_\Phi^{-1}\sim(T/\Delta)\tau_S^{-1} \gg \tau_S^{-1}$. We now evaluate τ_Φ^{-1}. Substituting $\varphi^{(-)}=\xi_p/\epsilon_p$ into (4.15) we see immediately that, as in (4.17) the region $\epsilon\sim T\gg\Delta$ is the important one, in the main terms in (4.15) the values $\epsilon'\sim\Delta\ll\epsilon$ are the important ones, so that we can expand in terms of the small parameter $\Delta/T \ll 1$. There remain then in (4.15) only the first and the last integrals in which the upper limits of integration can be replaced by $+\infty$. As a result we get

$$J_{ph}\left\{\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\right\}=-\frac{\pi\Delta}{16ms^2p_Fl_aT^2}\frac{\varepsilon_{\mathbf{p}}\xi_{\mathbf{p}}}{\operatorname{sh}(\varepsilon_{\mathbf{p}}/T)}. \quad (4.18)$$

Substituting (4.18) into (4.17) and integrating we find that

$$\frac{1}{\tau_\Phi}=\frac{1}{\tau_0}=\frac{7\zeta(3)T^2}{4ms^2p_Fl_a}. \quad (4.19)$$

Thus, in contrast to the fluctuation spectrum $\langle\delta\Delta^2\rangle_\omega$ the width of the fluctuation spectrum $\langle\Phi^2\rangle_\omega$ close to the transition point is independent of the distance to the transition point. We emphasize once again that our consideration becomes inapplicable in the immediate vicinity of the transition point, when $\Delta\to 0$.

At low temperatures we can find the asymptotic behavior of the spectral function for $\omega\tau\gg 1$. The calculations lead to the following result:[4)]

$$\langle\Phi^2\rangle_\omega=\frac{T^2}{\pi\Delta}\frac{N_n}{N}\frac{\partial\mu}{\partial N}\frac{1}{\omega^2\tau_\Phi^2}, \quad (4.20)$$

$$\frac{1}{\tau_\Phi}=C_\Phi\frac{T^2}{ms^2p_Fl_a}\left(\frac{T}{2\pi\Delta}\right)^{1/2}, \quad (4.21)$$

$$C_\Phi=\frac{\Gamma(5)}{2}\int_1^\infty\frac{dz}{z^{1/2}(z-1)^{1/2}}\zeta(5,z).$$

The correlator of the random external current can be found by substituting into the second term in (2.12) the correction to the distribution function proportional to $G_{\mathbf{p}}^{\omega\mathbf{q}}$, from (3.2). When $q = 0$

$$\langle J_i^{\mathrm{ext}}J_j^{\mathrm{ext}}\rangle_\omega=\frac{T}{\pi}\operatorname{Re}\sigma_{ij}(\omega). \quad (4.22)$$

We emphasize once again that due to the Meissner effect there are no fluctuations in the total current at $q = 0$ in the volume of the superconductor.

We study Eq. (4.22) for the case when the scattering by impurities is the deciding factor. In that case the collision operator has the relaxation time form

$$J_{\mathbf{p}}\psi_{\mathbf{p}}=\frac{1}{\tau_n}\frac{|\xi_{\mathbf{p}}|}{\varepsilon_{\mathbf{p}}}\psi_{\mathbf{p}},$$

where τ_n is the electron relaxation time in a normal conductor. Using (3.17) we easily obtain an expression for the correlators of the external currents in different limiting cases. At low temperatures

$$\langle J_i^{\mathrm{ext}}J_j^{\mathrm{ext}}\rangle_\omega=-\delta_{ij}\frac{\Delta}{\pi}\sigma_n(\omega)\exp\left[-\frac{\Delta}{T}+\frac{\Delta}{2T}\frac{(\omega\tau_n)^2}{1+(\omega\tau_n)^2}\right]$$

$$\times\operatorname{Ei}\left[-\frac{\Delta}{2T}\frac{(\omega\tau_n)^2}{1+(\omega\tau_n)^2}\right], \quad (4.23)$$

where $\sigma_n(\omega)=\sigma_0(1+\omega^2\tau_n^2)^{-1}$ is the high-frequency conductivity of a normal metal and σ_0 the static conductivity of a normal metal.

It follows from (4.23) that when $\Delta(\omega\tau_n)^2/T \ll 1$, i.e., at low frequencies, the correlator of the external currents has a logarithmic singularity

$$\langle J_i^{\mathrm{ext}}J_j^{\mathrm{ext}}\rangle_\omega=\delta_{ij}\frac{\Delta}{\pi}\sigma_0e^{-\Delta/T}\ln\frac{2T}{\Delta(\omega\tau_n)^2}. \quad (4.24)$$

When $\omega\tau_n \gtrsim 1$

$$\langle J_i^{\mathrm{ext}}J_j^{\mathrm{ext}}\rangle_\omega=\delta_{ij}\frac{2T}{\pi}\sigma_0e^{-\Delta/T}\frac{1}{(\omega\tau_n)^2}. \quad (4.25)$$

At high temperatures $\Delta \ll T$ the correlator of the external currents also has a logarithmic singularity at low frequencies:

$$\langle J_i^{\mathrm{ext}}J_j^{\mathrm{ext}}\rangle_\omega=\delta_{ij}\frac{T}{\pi}\sigma_n(\omega)\left\{1+\frac{\Delta}{2T}\left[\frac{1}{(1+\omega^2\tau_n^2)^{1/2}}\ln\frac{1+(\omega^2\tau_n^2+1)^{1/2}}{\omega\tau_n}-1\right]\right\}. \quad (4.26)$$

However, as $\Delta\to 0$ the logarithmic term vanishes and (4.26) changes to the usual expression for a normal metal.

In concluding this section we discuss the problem of the spatial correlation of the fluctuations. Of most interest, in our opinion, is the spatial correlation of the absolute magnitude of the order parameter. We emphasize that in our approximation $q \ll \Delta/v_F$ the equal-time correlator is proportional to $\delta(\mathbf{r}_1-\mathbf{r}_2)$. To study the spatial dispersion it is necessary to solve the equation

$$\left\{-i\omega+i\mathbf{qv}\frac{\xi}{\varepsilon}+J_{ph}+J_{im}\right\}\psi_{\mathbf{p}}=\frac{\Delta}{\varepsilon_{\mathbf{p}}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}. \quad (4.27)$$

We restrict ourselves to the high-temperature case, when $\Delta \ll T$, and we shall assume that the relaxation time due to impurities is much shorter than the relaxation time due to phonons. We restrict ourselves to the case $\omega\tau_n \ll 1$ and $ql_n \ll 1$, where l_n is the mean free path for scattering by impurities.

We average Eq. (4.27) over a surface of constant energy ϵ for a given sign of ξ. We shall indicate such an average by a bar over the quantity. We get

$$-i\omega\bar\psi_{\mathbf{p}}+i\overline{\mathbf{qv}\frac{\xi}{\varepsilon}\psi_{\mathbf{p}}}+\overline{J_{ph}\psi_{\mathbf{p}}}=\frac{\Delta}{\varepsilon}\frac{\partial n_0}{\partial\varepsilon}. \quad (4.28)$$

Subtracting (4.28) from (4.27) and inverting the operator

for impurity scattering we get a formal solution of Eq. (4.27):

$$\psi_{\mathbf{p}}=\bar\psi_{\mathbf{p}}+i\omega J_{im}^{-1}(\psi_{\mathbf{p}}-\bar\psi_{\mathbf{p}})$$

$$-iJ_{im}^{-1}\left(\mathbf{qv}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\psi_{\mathbf{p}}-\overline{\mathbf{qv}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\psi_{\mathbf{p}}}\right)-J_{im}^{-1}(J_{ph}\psi_{\mathbf{p}}-\overline{J_{ph}\psi_{\mathbf{p}}}). \quad (4.29)$$

The inverse operator J_{im}^{-1} in (4.29) is determined uniquely by the condition $\overline{J_{im}^{-1}}=0$.

We can iterate in Eq. (4.29) in terms of the parameters $\omega\tau_n \ll 1$ and $ql_n \ll 1$:

$$\psi_{\mathbf{p}}^{(1)}=\bar\psi_{\mathbf{p}}-iJ_{im}^{-1}\mathbf{qv}\frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\bar\psi_{\mathbf{p}}. \quad (4.30)$$

Substituting (4.30) into (4.28) we get

$$\left(-i\omega+\frac{|\xi_{\mathbf{p}}|}{\varepsilon_{\mathbf{p}}}Dq^2+J_{ph}\right)\bar\psi_{\mathbf{p}}=\frac{\Delta}{\varepsilon_{\mathbf{p}}}\frac{\partial n_0}{\partial\varepsilon_{\mathbf{p}}}, \quad (4.31)$$

where $D=v_F^2\tau_n/3$ is the diffusion coefficient in a normal metal.

When $\Delta/T \ll 1$ we showed above that the phonon operator reduces to the relaxation time and we get for ψ_p the following expression:

$$\bar\psi_{\mathbf{p}}=-\frac{\Delta}{4T}\frac{1}{\varepsilon_{\mathbf{p}}}\left(-i\omega+Dq^2\frac{|\xi_{\mathbf{p}}|}{\varepsilon_{\mathbf{p}}}+\frac{1}{\tau_0}\right)^{-1}. \quad (4.32)$$

Substituting (4.32) into Eq. (3.4) for $\gamma_{q\omega}$ we get

$$\gamma_{\mathbf{q}\omega}=1-i\omega\tau_{\mathbf{q}\omega}/(1-i\omega\tau_0). \quad (4.33)$$

We used here the notation

$$\tau_{\mathbf{q}\omega}=-i\frac{N}{N_s}\frac{\Delta}{2T}\tau_0\left[1-\left(\frac{Dq^2\tau_0}{1-i\omega\tau_0}\right)^2\right]^{-1/2}$$

$$\times\ln\left\{\frac{Dq^2\tau_0}{1-i\omega\tau_0}+i\left[1-\left(\frac{Dq^2\tau_0}{1-i\omega\tau_0}\right)^2\right]^{1/2}\right\}. \quad (4.34)$$

Then, from (3.5),

$$\langle\delta\Delta^2\rangle_{\omega\mathbf{q}}=\frac{T}{\pi}\frac{\partial\mu}{\partial N}\frac{N}{N_s}\frac{\operatorname{Re}\tau_{\mathbf{q}\omega}-\omega\tau_0\operatorname{Im}\tau_{\mathbf{q}\omega}}{1+\omega^2|\tau_0+\tau_{\mathbf{q}\omega}|^2}. \quad (4.35)$$

We find $\tau_{\omega q}$ for different limiting cases. We consider the frequency range $\omega\tau_0 \ll 1$. Then $\operatorname{Im}\tau_{q\omega}\sim\omega\tau_0$ and $\tau_{q\omega}$ is in this approximation a purely real quantity, independent of ω. If $Dq^2\tau_0 < 1$, we have

$$\frac{1}{\tau_{\mathbf{q}\omega}}=\frac{1}{\tau_\Delta}\frac{\pi}{2}\frac{[1-(Dq^2\tau_0)^2]^{1/2}}{\arccos Dq^2\tau_0}, \quad (4.36)$$

where $1/\tau_\Delta = N_S4T/\tau_0N\pi\Delta$ is the reciprocal of the time of the uniform relaxation of the absolute magnitude of the order parameter, (4.11), when $\Delta \ll T$.

If $Dq^2\tau_0 \ll 1$, we have

$$\frac{1}{\tau_{\mathbf{q}\omega}}=\frac{1}{\tau_\Delta}+Dq^2\frac{8T}{\pi^2\Delta}\frac{N_s}{N}\equiv\frac{1}{\tau_\Delta}+D_\Delta q^2. \quad (4.37)$$

As $Dq^2\tau_0\to 1$

$$\frac{1}{\tau_{\mathbf{q}\omega}}=\frac{\pi}{2}\frac{1}{\tau_\Delta}. \quad (4.38)$$

If $Dq^2\tau_0 > 1$, we have

$$\frac{1}{\tau_{\mathbf{q}\omega}}=\frac{1}{\tau_\Delta}\frac{\pi}{2}\frac{((Dq^2\tau_0)^2-1)^{1/2}}{\ln[Dq^2\tau_0+((Dq^2\tau_0)^2-1)^{1/2}]} \quad (4.39)$$

Thus, in the whole range considered for changes in the wavevectors $\tau_{q\omega}\gg\tau_0$ and $\langle\delta\Delta^2\rangle_{\omega q}$ has thus finally the form

$$\langle\delta\Delta^2\rangle_{\omega\mathbf{q}}=\frac{T}{\pi}\frac{\partial\mu}{\partial N}\frac{N}{N_s}\frac{\tau_{\mathbf{q}\omega}}{1+\omega^2\tau_{\mathbf{q}\omega}^2} \quad (4.40)$$

From the expressions given here for $\tau_{q\omega}$ it is clear that spatially non-uniform relaxation has a diffusion character only when $Dq^2\tau_0 \ll 1$ with a diffusion coefficient which vanishes as $T\to T_c$: $D_\Delta\propto(T_c-T)^{1/2}$.

5. MAGNETIC FLUX FLUCTUATIONS IN A SUPERCONDUCTING RING. LINE WIDTH OF THE JOSEPHSON EFFECT

We consider a superconducting cylinder of radius R_0, height d, and thickness a, in which an integral number of flux quanta are included. We evaluate the integral of p_{si} over a closed contour through the thickness of the ring:

$$\oint\mathbf{p}_s\,d\mathbf{l}=-\frac{\phi-\bar\phi}{\phi_0}\pi\hbar=-\frac{\delta\phi}{\phi_0}\pi\hbar, \quad (5.1)$$

where ϕ is the complete magnetic flux, including its fluctuations and $\bar\phi$ is the average flux, equal to $n\phi_0$ ($\phi_0=\pi\hbar c/e$).

We evaluate the correlator of the magnetic flux fluctuations through the given contour:

$$\langle\delta\phi(\mathbf{r}_\perp)\delta\phi(\mathbf{r}_\perp)\rangle_\omega=\left(\frac{\phi_0}{\pi\hbar}\right)^2 2\pi R_0\langle p_s(\mathbf{r}_\perp)p_s(\mathbf{r}_\perp)\rangle_{\omega q_{\parallel}=0}, \quad (5.2)$$

where $q_{\parallel}$ is the component of the wavevector along the direction of the tangent to the contour, and $\mathbf{r}_\perp$ a vector at right angles to the contour. We determine first the integral intensity of the magnetic flux fluctuations. If all dimensions of the cylinder are much larger than the penetration depth of the magnetic field, we get, using (4.4) and integrating over the two-dimensional $q_\perp$,

$$\langle\delta\phi^2\rangle=2\pi R_0T\ln(N/N_s). \quad (5.3)$$

For a thin-walled cylinder with dimensions $a < \lambda_L$

$$\langle\delta\phi^2\rangle=(2\pi)^2R_0T\frac{\lambda_L}{a}\left(1-\left(\frac{N_s}{N}\right)^{1/2}\right) \quad (5.4)$$

and, finally, for a thin ring with a height small compared with λ_L

$$\langle\delta\phi^2\rangle=8\pi^2TR_0\lambda_L^2N_n/adN. \quad (5.5)$$

Burgess[10] has earlier obtained Eq. (5.5) for a thin ring. It follows from (5.2) that in a thin ring the frequency spectrum of the magnetic flux fluctuations is determined by different-time correlators of the superconducting momentum at $q = 0$, i.e., by Eq. (3.15).

The magnetic flux fluctuations in a cylinder thus have a classical nature and quantum limitations, imposed on the phase of the wavefunction in the ring, do not affect their dynamics, as they are connected with random processes in the normal component of the superconductor.[5)]

The fluctuations in the potential Φ can determine the line width of the Josephson radiation. Kulik[11] was the first to discuss the effect of the fluctuations in the phase of the order parameter on the radiation line width; he considered the problem of the radiation line width near the transition point in superconductors with paramagnetic impurities using the non-stationary Ginzburg-Landau equation. Knowing the fluctuation spectrum $\langle\Phi^2\rangle_\omega$ we can express the line shape of the radiation from a point contact in terms of the spectral parameters. We shall assume that the potentials Φ in the two superconductors which form a weak link fluctuate independently. This is justified as we take the whole of the weak link into account using perturbation theory. The Josephson current is thus

$$J(t)=J_c\sin\varphi(t), \quad (5.6)$$

Where J_c is the critical current, and the phase difference at the transition has the form

$$\varphi(t)=\int_{t_0}^{t} dt'[2eV_0+2eV(t')+2\Phi_1(t')-2\Phi_2(t')], \qquad (5.7)$$

where V_0 is the potential difference applied to the transition, and $V(t)$ the fluctuation potential difference connected with the finite resistance of the tunnel connect: $\langle V^2(t)\rangle_\omega = TR_T/\pi$. When writing down Eq. (5.7) we used the gauge with $\varphi = 0$ inside the superconductors. This is the only gauge for which the fluctuations in the phase difference (i.e., the fluctuations in the difference in the chemical potentials) and the fluctuations in the difference of the scalar potentials $V(t)$ are statistically independent.[11]

We follow Kulik and Yanson[12] and introduce the spectral function

$$I(\omega)=\frac{2}{\pi}\int_{-\infty}^{+\infty} e^{i\omega\tau}\langle \sin\varphi(t)\sin\varphi(t+\tau)\rangle d\tau. \qquad (5.8)$$

After that the calculation is completely equivalent to the one given in[12]. We note merely that as the line width is much less than $1/\tau_{ph}$, which is the quantity which determines the dispersion of the fluctuation spectrum $\langle \Phi_i^2\rangle_\omega$, the linewidth is determined by $\langle \Phi_i^2\rangle_{\omega=0}$.

Simple calculations show that in that case

$$I(\omega)=\frac{1}{\pi}\frac{\Gamma}{(\omega-\omega_0)^2+\Gamma^2}, \qquad (5.9)$$

$$\Gamma=4e^2R_TT/\hbar^2+4\pi[\langle \Phi_1^2\rangle_0+\langle \Phi_2^2\rangle_0]. \qquad (5.10)$$

If the superconductors are the same, we get, using (4.16) near the transition point for a point contact

$$\Gamma=\frac{4T}{\hbar^2}\left\{e^2R_T+\frac{1}{v_0}\frac{8T}{\pi\Delta}\frac{\partial\mu}{\partial N}\tau_\Phi\right\} \qquad (5.11)$$

(v_0 is the volume of the superconductor). Estimates show that Γ may be of the order of 10^3 to 10^4 Hz.

Taking the phase fluctuations into account leads to the fact that in the expression describing the current-voltage characteristics (see, e.g.,[12]) there occurs not the temperature, but T^*, an effective noise temperature for the contact:

$$T^*=\hbar^2\Gamma/4e^2R_T. \qquad (5.12)$$

We note finally that we can also measure the spectral function of the fluctuations in the complex order parameter directly, by measuring the excess currents in a Josephson transition.[13]

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1)In a state far from equilibrium the criterion may be more rigid: $\omega \ll \bar{\epsilon}$, where $\bar{\epsilon}$ is a characteristic scale for changes in the distribution function of the excitations. Everywhere in the paper $k = \hbar = 1$, and the volume is also taken to be unity.

2)We note that the eigenvalues of the operator $\hat{L}_p$ are not equal to -1, and therefore the solution of the equation $(1 + \hat{L}_p)\psi_p = \varphi_p$ in the form $\psi_p = (1 + \hat{L}_p)^{-1}\varphi_p$ exists and is unambiguous.

3)Below we discuss the expression for the correlators of external random currents in a superconductor.

4)At low temperatures the fluctuations in the quantities Δ and Φ in the frequency range of interest remain classical (Φ is the Debye temperature):

$$\omega \lesssim \tau_\Delta^{-1} \sim \tau_\Phi^{-1} \sim T^3\Theta^{-2}(T/\Delta)^{1/2} \ll T$$

5)We emphasize that the fluctuations considered are nothing but the quasi-static magnetic field fluctuations.

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Description of the Coulomb interaction in the theory of superconductivity and calculation of T_c

L. B. Dubovskiĭ and A. N. Kozlov

I. V. Kurchatov Atomic Energy Institute
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Zh. Eksp. Teor. Fiz. **68**, 2224–2235, June 1975

A method is proposed for solving the Éliashberg equation for anisotropic superconductors in the temperature technique at $T = T_c$. The solution is found by successive calculation of the terms of order $\lambda\ln(\omega_D/T_c)$, λ, λ^2,..., where $\lambda = \lambda_0/(1+\lambda_0)$ is the renormalized electron-phonon coupling constant. A consistent way of taking the Coulomb interaction of the electrons into account is described and a definition of the Coulomb pseudopotential in the anisotropic model is given. A general expression for T_c, including corrections of order λ, is given. The dependence of the effective mass on the energy gives a contribution to the corrections of order λ^2 and higher. For the Einstein model T_c is calculated to order λ^2. For a model in which the phonon spectrum consists of two Einstein peaks the equations are solved numerically and the dependence of T_c of the frequency ω_1 of one of the peaks is determined. It is shown that as $\omega_1 \to 0$ this peak gives a finite contribution to T_c if $\lambda_1 \sim \omega_1^{-2}$. If $\lambda_1 \sim \omega_1^{-2+\nu}$, where $\nu > 0$, the contribution from the low-frequency peak vanishes in the limit $\omega_1 \to 0$.

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There are a large number of papers devoted to deriving an approximate analytic formula for the T_c of strong-coupling superconductors. (Such attempts are undertaken with the purpose of going beyond the framework of the BCS approximation, in order to describe the experimental situation.) The best-known is the empirical formula of McMillan[1]:

$$T_c=\frac{\Theta}{1.45}\exp\left[-\frac{1.04(1+\lambda_0)}{\lambda_0-\mu^*(1+0.62\lambda_0)}\right], \qquad (1)$$

which was obtained by fitting the results of a numerical solution of the Éliashberg equation to a simple analytic form[2]. An electron-phonon interaction function $\alpha^2(\omega)F(\omega)$ extracted from tunneling measurements on niobium was used. Here,

$$\lambda_0=2\int_0^\infty \alpha^2(\omega)F(\omega)d\omega/\omega$$

is the electron-phonon coupling constant, μ^* is the Coulomb pseudo-potential and Θ is the Debye temperature. Although formula (1) is valid just for niobium, it is often applied to any superconducting metal, even though in such cases there are no adequate reasons for preferring the McMillan formula to the BCS formula. The attempts to obtain an empirical formula of the same type for other superconductors are well-known[3-7]. These formulas differ from (1) in the numerical coefficients, and all of them are nonuniversal, since they pertain to superconductors with specific phonon spectrum. In recent papers[8-11], expressions of a more general type for T_c are derived. In these formulas, functionals of $\alpha^2(\omega)F(\omega)$ appear in the role of the numerical coefficients. To derive these formulas[10,11] one uses approximate solutions of the Éliashberg equation, obtained, e.g., by substituting a step function or the Morel-Anderson function[12] as a first approximation for the gap function $\Delta(\omega)$. No attempts are made to estimate the accuracy of the approximation, inasmuch as the procedure turns out to be extremely cumbersome even in the first stage. In essence, such a procedure corresponds to determining the first correction in the constant $\lambda = \lambda_0/(1 + \lambda_0)$ to the exponential factor $\lambda - \mu^*$ and is valid for sufficiently small λ_0. In particular, it cannot be used to elucidate the question of the influence of the low-frequency phonon peaks on the magnitude of T_c if the coupling constant for the coupling with the low-frequency mode is large. The erroneous result obtained in[10,11] concerning the role of the soft phonon modes is connected with this fact.

It should be noted that the solution of the Éliashberg equation, like the form of the equation itself, is considerably simplified in the temperature technique[13], since in this representation all quantities are real and the kernel has no singularities. Formulas for T_c going beyond the framework of the first approximation were obtained in the temperature technique in[14] for phonon spectra with one and two Einstein modes.

In this paper we shall describe a regular method for solving the Éliashberg equation in the temperature technique for the general case of an anisotropic superconductor with an arbitrary phonon spectrum. The solution is represented in the form of a series in powers of the electron-phonon interaction. We shall derive a strong-coupling formula for T_c in the anisotropic model and shall calculate T_c in certain particular cases; we shall also consider the question of the effect on T_c of the low-frequency phonon peaks.

1. DERIVATION OF THE EQUATIONS

The Coulomb interaction of the electrons in a superconductor has been taken into account most consistently by Batyev[15] in the framework of the isotropic model. We shall formulate the corresponding result, having modified it for a variant of the temperature technique, without assuming that the interaction is isotropic. The equation for the determination of T_c has the form

$$\Delta_{np}=-T\sum_m\int\frac{d\mathbf{p}'}{(2\pi)^3}(D^{nm}_{\mathbf{pp}'}+K^{nm}_{\mathbf{pp}'})R_{m\mathbf{p}'}\Delta_{m\mathbf{p}'}, \qquad (2)$$

where

$$D^{nm}_{\mathbf{pp}'}=-\sum_j|g^j_{\mathbf{pp}'}|^2\frac{\omega^2_{j,\mathbf{p}-\mathbf{p}'}}{\omega^2_{j,\mathbf{p}-\mathbf{p}'}+\omega^2_{n-m}}; \qquad (3)$$

$g^j_{pp'}$ is the total (with allowance for all the Coulomb corrections) amplitude for scattering of an electron by a phonon with the j-th polarization; $K^{nm}_{pp'}$ is the total Coulomb amplitude; $R_{np} = G_{np}G_{-n,-p}$ where G_{np} is the normal electron Green function, which near the Fermi surface has the form