Statistics of mesoscopic fluctuations and instability of one-parameter scaling

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The statistics of the fluctuations of the conductance G and of the density of states ν in an ensemble of small samples having identical macroscopic characteristics is investigated in a field-theoretical approach using the nonlinear σ-model. It is shown that at not too large deviations from the mean the fluctuations are described within the framework of one-parameter scaling (the scaling parameter is the average conductance $\langle G \rangle$). The validity of this statement is based on the Einstein relation and on the continuity equation. In the metallic region ($G \gg e^2/\hbar$) fluctuations of G and ν that are not too large are described by a normal Gaussian distribution. In the region of the Anderson transition ($G \lesssim e^2/\hbar$), however, one-parameter scaling leads to fluctuations that deviate greatly from Gaussian. The probability of large fluctuations of G and ν is much higher than Gaussian even in the metal region: the tails of the G and ν distribution functions turn out to be logarithmically normal. The presence of non-Gaussian tails is due to the instability of the standard σ model when account is taken of the additional vertices that arise in a consistent description of the high fluctuation moments. To consider each of the moments it is necessary to introduce independent scaling parameters whose number increases with increasing order of the moments, so that the one-parameter scaling breaks down.

§1. INTRODUCTION

Scaling theory in Anderson's localization problem was ultimately formulated by Abrahams, Anderson, Licciardello, and Ramakrishnan,[1] and has long been universally accepted. This theory is based on the assumption that the only significant scaling parameter of a sample is its residual conductance G. This means that the change of the dimensionless conductance of a d-dimensional cube due to the change of its dimension L is described by the equation

$$\frac{dg}{d\ln L} = \beta(g), \quad g \propto \frac{G\hbar}{e^2}. \tag{1}$$

Perturbation-theory calculations[2] in the region of metallic conductivity ($G \gg e^2/\hbar$) in conjuction with the renormalizability hypothesis[1] yielded the following result for the Gell-Mann–Low function:

$$\beta(g) = (d-2)g - 1 + O(g^{-1}). \tag{2}$$

The result (2) was confirmed by direct calculation of $\beta(g)$ in a field-theoretical approach using the nonlinear σ model proposed in localization theory by Wegner[3] and further developed by others.[4–9] It follows from (1) and (2) that at $d = 2$ the dependence of the conductance on the scale is given by

$$g^{-1} = \frac{g_0^{-1}}{1 - g_0^{-1}\ln L/l}, \tag{3}$$

where g_0 is the conductance of a d-dimensional cube whose edge equals the mean free path l at $T = 0$. Equation (3) is valid down to $g \sim 1$. Extrapolation of this expression to the region of small g is in fact the only microscopic confirmation of one of the main qualitative conclusions of localization theory,[1] viz., that localized states exist at $d = 2$ for arbitrarily weak disorder ($g_0 \to \infty$). Doubts were cast, however, on the microscopic validity of one-parameter scaling.

The results (1)–(3) are valid for conductance averaged over realizations of the random potential (over the disposition of the impurities). It was made clear in a number of recent papers that the characteristics of individual samples cannot be determined from averaged quantities. This forces us again to check whether the one-parameter scaling (1)–(3) can be applied to the localization problem.

The concept of average conductance is untenable because the conductance of an ensemble of sample having identical macroscopic characteristics fluctuates noticeably from sample to sample[13–34] (see also Refs. 35–37 and the bibliographies therein). It was recently established[24,25] that even in the region of weak localization ($g \gg 1$) these fluctuations are anomalously large:

$$\langle (\delta G)^2 \rangle \sim (e^2/\hbar)^2 \tag{4}$$

(here $\delta G = G - \langle G \rangle$ and $\langle \ldots \rangle$ denotes averaging over the realizations of the random potential, i.e., over the entire ensemble of samples). The relative size of these fluctuations is $\delta G / G \propto L^{2-d}$, i.e., it decreases more slowly than $L^{-d/2}$ even at $d = 3$.

Of course, the actual sample investigated in each experiment has a uniquely defined impurity arrangement in each experiment. If the fluctuations (4) from sample to sample are appreciable, a macroscopic approach based on the calculation of $\langle G \rangle$ is inadequate. On the other hand, a microscopic calculation of non-averaged characteristics is neither realistic nor very instructive.

Under these conditions it is natural to regard the conductance as a random quantity. Besides mere calculation of $\langle G \rangle$, a study must be made of the entire distribution function of the conductances in an ensemble of samples having identical macroscopic characteristics. This approach was named mesoscopic. The fluctuations of the conductance and of other sample characteristics over the ensemble are customarily called mesoscopic. At low but finite temperature T, these fluctuations are significant if the sample dimension satisfies the condition

$$l \ll L \lesssim L_T,\ L_\varphi, \tag{5}$$

where $L_T = (D\hbar/T)$ is the diffusion length within the time $\hbar/T$, and $L_\varphi = (D\tau_\varphi)^{1/2}$ is the phase-coherence-loss length of the electron wave function (D is the phase-coherence-loss time in inelastic scattering) Samples with dimensions (5) are now called mesoscopic.

The mesoscopic fluctuations in the metallic-conductivity region ($g > 1$) have manifested themselves in the form of reproducible aperiodic oscillations of G as functions of the magnetic field H[38–43] or of chemical potential ε_F.[44] The assumption that the conductance-fluctuation statistics of a given sample, as a function of H or ε_F, can be identified with the statistics of the fluctuations in an ensemble of samples was set forth in Ref. 25 and is called the ergodic hypotheses.

We have discussed so far only the mean squared fluctuations $\langle \delta(G)^2 \rangle$ and only in the lowest perturbation-theory order in g_0^{-1}. It would be desirable, however, to have theory similar to scaling theory,[1] capable of describing not only $\langle G \rangle$ but also the mesoscopic fluctuations. To develop such a theory we must be able, first, to calculate the high moments of the fluctuations and, second, sum the corrections to them in all orders of perturbation theory in g_0^{-1}. All this will be done in the present paper using a field-theoretical approach.

§2. PRINCIPAL RESULTS

The first problem to be solved is that of the dependence of $\langle (\delta G)^2 \rangle$ on the system dimension L with increase of L. This problem cannot be solved for $d \leqslant 2$ within the framework of perturbation theory, since (4) is subject to corrections that diverge as $L \to \infty$. These corrections, however, can be quite easily summed if the one-parameter scaling is valid.

The irreducible fluctuation moments $\langle (\delta G)^2 \rangle_c$ can indeed be described within the framework of one-parameter scaling if n is not too large. This statement is based on the Einstein relation between the conductance G and the diffusion coefficient D, and on the continuity equation. One-parameter scaling means that summation of all the divergent (logarithmically at $d = 2$) corrections to these moments reduces to replacing the classical conductance g_0 in all the nonlogarithmic expressions by its renormalized value $g(L)$ [Eq. (3)].[1)]

The conductance variance is independent of g_0 [see (4)]. A dependence on g sets in therefore only when higher-order nonlogarithmic corrections are taken into account. Therefore $\langle (\delta G)^2 \rangle$ remains universal in the metallic-conductivity region ($g \gg 1$) accurate to powers of g^{-1}. The mean value of the conductance, on the other hand, changes in this region by a factor g_0, with $g_0 \gg 1$ (Fig. 1a).

We consider also the mesoscopic fluctuations, investigated in Ref. 27, of the density of states. The mean squared fluctuations $\langle (\delta\nu)^2 \rangle^{1/2}$ also change by a factor g_0, increasing from $\sim \nu_0 g_0^{-1}$ (Ref. 27) at $L \sim l$ to $\sim \nu_0$ at $L \sim L_0 = l \times \exp g_0$, where ν_0 is the mean density of states and is known to be independent of disorder (Fig. 1b).

We shall show also that the higher fluctuation moments of the conductance and of the density of states increase with increasing L. At $n < g_0$ we obtain for the cumulants $\langle (\delta G)^n \rangle_0$ (the irreducible mean values of $(\delta G)^n$) the expression

$$\langle (\delta G)^n \rangle_c \propto g^{2-n} \langle (\delta G)^2 \rangle^{n/2} \propto g^{2-n} (e^2/\hbar)^n, \tag{6}$$

where $g(L)$ is given by Eq. (3). A similar relation holds also for the moments of the density of states.

It can be seen from (6) that in the metallic conductivity region $g \gg 1$ the higher ($n > 2$) cumulants are small. At large g and and not too large δg, the $f(\delta g)$ distribution is close to normal Gaussian. It follows at the same time from (3) and (5) that the high cumulants increase with increasing L and that $\langle (\delta G)^n \rangle_{c^{1/n}}$ becomes of the order of $\langle G \rangle$ at $g \sim 1$ (Fig. 1a). This means that the distribution $f(\delta G)$, as well as the distribution $f(\delta\nu)$, deviates strongly from Gaussian in this region.

We show also in this paper that even at $g \gg 1$ the tail of the distribution function differs drastically from Gaussian: the normal distribution

$$f(\delta g) \propto \exp[-(\delta g)^2]$$

is replaced at $\delta g \gtrsim \Delta$ by the logarithmic normal distribution

$$f(\delta g) \propto \exp\left[-\frac{1}{8u}\ln^2(\delta g L^2/l^2)\right]. \tag{7}$$

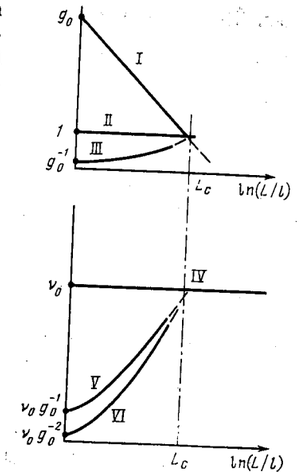

FIG. 1. Irreducible moments of the fluctuations of the conductance (a) and of the density of states (b) vs the system dimensions L:I—average conductance g; II—variance of conductance $\langle (\delta g)^2 \rangle^{1/2}$; III—higher fluctuation moments $\langle (\delta g)^n \rangle_{c^{1/n}}$, at $n < g_0$; IV—average density of states ν_0; V—variance of density of states $\langle (\delta\nu)^2 \rangle^{1/2}$; VI—higher fluctuation moments $\langle (\delta\nu)^n \rangle_{c^{1/n}}$ at $n < g_0$.

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A similar asymptotic form is obtained also for the distribution function of the density of states by making the changes $u \to u/2$ and $\delta g \to g\delta\nu/\nu_0$. Here

$$\Delta^2 = \frac{\ln L/l}{u}, \qquad u \sim \begin{cases} g_a^{-1}L/a, & d=1 \\ g_0^{-1}\ln L/l, & d=2. \\ g_0^{-1}, & d=3 \end{cases} \tag{7a}$$

In the quasi-one-dimensional case we have $g_a \equiv g(L=a)$, where a is the transverse dimension of the wire. It is remarkable and patently not fortuitous that the distribution (7), obtained by us for $g>1$ and for arbitrary dimensionality d, coincides with the exact distribution obtained in Refs. 17 and 50 for $d=1$ and $g<1$.

It is of fundamental importance that the asymptotic depends on the unrenormalized value of the conductance g. This is direct evidence of violation of one-parameter scaling. The point is that to describe the moments $\langle(\delta G)^n\rangle_c$ at $n \gtrsim \Delta^2 \gg 1$, which determine in fact the tails of the distribution function (7), it is necessary to take into account additional scaling parameters (that are independent of g).

Violation of the one-parameter scaling theory, due to appearance of additional scaling parameters, was observed in Ref. 10 in connection with the question of the frequency dependence of D. The diffusion-equation corrections that arise at finite distances and at finite times are known to be manifested in the dependence of the diffusion coefficient D on the frequency ω and on the wave vector $\mathbf{q}$. In the classical case this dependence can be represented in the form

$$D(\mathbf{q},\omega) = D + \sum_{n,m} \gamma_{n,m}(\omega\tau)^n (ql)^{2m}, \tag{8}$$

where $\tau = l/v_F$ is the free-path time.

Of course, at $\omega\tau \ll 1$ and $ql \ll 1$ these corrections are small compared with the quantum corrections to the diffusion coefficient (the latter are logarithmic at $d=2$). The quantum corrections to the coefficient $\gamma_{n,0}$, however, cause it to increase (when the scale or frequency is changed) in proportion to $(g_0/g)^{n^2}$ (Ref. 10), so that at $n > \Delta^2 \sim g_0$ the corrections (8) become significant. The coefficients γ are then described by a number of additional scaling parameters. This makes, in particular, the Anderson transition much more complicated: not only does the diffusion coefficient decrease in the region $g \lesssim 1$, but the equation that describes the electron-density fluctuations in this region also differs drastically from the diffusion equation.

We consider in the present paper only the homogeneous static conductivity ($q = \omega = 0$). The additional scaling parameters needed for the description of the higher moments of the mesoscopic fluctuations turn out then to be analogous to the coefficients $\gamma_{n,0}$ in Eq. (8), and the role of the bare small parameter $(\omega\tau)^n$ is assumed by the ratio $(l/L)^{2n}$.

The relation $\ln \gamma_{n,0} \propto n^2$ found in Ref. 10 turns out to be universal. We show in the present paper that an analogous increase obtains for the contributions made to $\langle(\delta g)^n\rangle_c$ and $\langle(\delta\nu)^n\rangle_c$, by the additional scaling parameters. Thus, the contribution to $\langle(\delta g)^n\rangle_c$ in excess of (6) is

$$\langle(\delta g)^n\rangle_c^{add} \propto \left(\frac{l}{L}\right)^{2n}\left(\frac{g_0}{g}\right)^{2n^2}. \tag{9}$$

This yields in the perturbation-theory region $g_0^{-1}\ln L/l \ll 1$

$$\langle(\delta g)^n\rangle_c^{add} \propto (l/L)^{2n(1-n/g_0)}. \tag{10}$$

This quantity decreases at $n < g_0$ with increase of the dimension L, so that the cumulant $\langle(\delta g)^n\rangle_c$ is determined for such n by the "normal" one-parameter contribution (6), and the normal Gaussian distribution is valid at sufficiently small δg. On the other hand, for moments of the order of $n \gtrsim g_0$ the dominant role is assumed by the additional contribution (9), which increases with the sample dimension L, and it is this which leads to the distribution function at $\delta g \gtrsim \Delta \propto g_0^{1/2}$.

The plan of the paper is the following: In §3, starting from the usual model of noninteracting electrons in a random potential, we obtain a functional representation of the mesoscopic fluctuations. In §4 is derived an analogous representation for the fluctuations of the density of states. In §5, with the mean squared fluctuations as the example, we show how to develop a regular perturbation theory by using the obtained functional representation. In §6, on the basis of the Einstein relation and the continuity equation, we prove certain exact relations for the renormalization-group charges of the theory, which lead, in particular, to the results illustrated in Fig. 1. In §7 are considered the "normal" contributions to the higher fluctuation moments, which lead to Eq. (6). In §8 we study the contributions to the cumulants described by the additional scaling parameters that lead to (9). In §9 is derived the asymptotic of the distribution function (7). A preliminary report of the results of the present paper was published in Ref. 45, where the ergodic hypothesis was proved.

§3. FUNCTIONAL REPRESENTATION FOR CONDUCTANCE FLUCTUATIONS

In the field-theoretical description of the kinetics of non-interacting electrons it is customary to use a functional representation of a density-density correlator[3-9] from which the diffusion coefficient can be determined. To study the mesoscopic fluctuations of the conductivity it is necessary to use a functional representation of the electron-electron correlator.[46] We use as the basis a field-theoretical formulation with functional integration over Grassman (anticommuting) conjugate fields $\chi(\mathbf{r})$ and $\varkappa(\mathbf{r})$ (Ref. 47), which is described in Ref. 4 (see also Refs. 10 and 48). The expression for the conductance $G_{\alpha\beta}$ (α and β are vector indices in 2-dimensional space) can be written in the form

$$G_{\alpha\beta} = -\frac{1}{\pi}\langle j_\alpha^{12} j_\beta^{21}\rangle_{\chi\varkappa}, \tag{11}$$

which is the functional form of the Kubo formula.[2] The current j^{ab} is given here by[3)]

$$\mathbf{j}^{ab} = \frac{ie}{2m_e L}\int \{\chi^a(\mathbf{r})\nabla\varkappa^b(\mathbf{r}) - \nabla\chi^a(\mathbf{r})\varkappa^b(\mathbf{r})\}\, d^d r. \tag{12}$$

The angle brackets $\langle\ldots\rangle_{\chi\varkappa}$ in (11) denote functional averaging with weight $\exp(i\mathcal{S})$:

$$\langle\ldots\rangle_{\chi\varkappa} = \frac{\int \mathcal{D}\chi\mathcal{D}\varkappa(\ldots)e^{i\mathcal{S}}}{\int \mathcal{D}\chi\mathcal{D}\varkappa e^{i\mathcal{S}}} \tag{13}$$

The functional of the action $\mathcal{S}$ is defined as

$$\mathcal{S} = \int \chi^a(\mathbf{r})\{(\varepsilon_F - \mathcal{H})\delta^{ab} + i\eta(\sigma_z)^{ab}\}\varkappa^b(\mathbf{r})\, d^d r \tag{14}$$

($\eta \to +0$, and ε_F is the Fermi energy), where $\mathcal{H}$ is the single-particle electron Hamiltonian

$$\mathcal{H} = -\frac{1}{2m_e}\nabla^2 + U(\mathbf{r}), \tag{15}$$

and $U(\mathbf{r})$ is a Gaussian random potential with a correlator

$$\langle U(\mathbf{r})U(\mathbf{r}')\rangle = \frac{1}{\pi\nu_0\tau}\delta(\mathbf{r}-\mathbf{r}'). \tag{16}$$

Here ν_0 is the average single-spin density of states, $\tau = l/v_F$, and l is the mean free path.

The superscripts a and b in (11)–(13) indicate that the fields $\chi^a(\mathbf{r})$ and $\varkappa^a(\mathbf{r})$ stem from the retarded $\mathscr{G}^r$ ($a=1$) or advanced $\mathscr{G}^a$ ($a=2$) Green's function of the electron in the usual representation of the conductance in terms of exact Green's functions in a random field $U(\mathbf{r})$ (see Fig. 2).

For subsequent averaging over the realizations of the potential we assign to each Green's function in Fig. 2 a replica index A that runs through the values 1 to N and transform, following Ref. 4, to "spinor" fields

$$\psi = \frac{1}{\sqrt{2}}\begin{pmatrix}\chi_A^a \\ \varkappa_A^a\end{pmatrix} \qquad \overline{\psi} = \frac{1}{\sqrt{2}}(-\varkappa_A^a, \chi_A^a). \tag{17}$$

In this notation, expression (11) is transformed into

$$G_{\alpha\beta} = -\frac{4}{\pi N^2}\langle \mathrm{Tr}(C^+ J_\alpha C^- J_\beta)\rangle_{\chi\varkappa}, \tag{18}$$

where the current matrix $\mathbf{J}$ is given by

$$\mathbf{J} = -\frac{ie}{2m_e L}\int \{\psi(\mathbf{r})\otimes\nabla\overline{\psi}(\mathbf{r}) - \nabla\psi(\mathbf{r})\otimes\overline{\psi}(\mathbf{r})\}\, d^d r. \tag{19}$$

In Eq. (18), $C^\pm$ are the projectors:

$$C^\pm = {}^1\!/_4(\delta^{ab} \pm (\sigma_z)^{ab})\otimes(\tau_0 + i\tau_3)\otimes\delta_{AB}. \tag{20}$$

The first factor in (20) leaves in (18) the current components that are off-diagonal in the upper indices in accordance with (11). The second (quaternion) factor that acts in "spinor" space retains in the current (19) the products $\chi\varkappa$ that enter in (12), and sets the products $\chi\chi$ and $\varkappa\varkappa$ equal to zero. Here τ_i are quaternion units $\tau_0 = I$ and $\tau_{1,2,3} = i\sigma_{x,y,z}$ (σ are Pauli matrices).

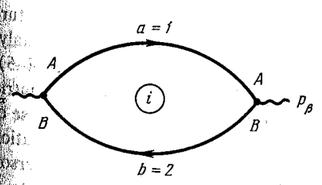

FIG. 2 Arrangement of the indices in the current loop: $A, B = 1, 2, \ldots, N$—replica indices; i—index numbering the current loops; α, β—vector indices; a, b—indices of the retarded ($a=1$) and advanced ($a=2$) Green's functions.

The expression for the nth power of the conductance is similar to (18):

$$(G)^n = \left(-\frac{4}{\pi N^2}\right)^n \left\langle \prod_{i=1}^{n} \mathrm{Tr}(C^+ J_i C^- J_i)\right\rangle_{\chi\varkappa}. \tag{21}$$

Each field $\overline{\psi}$ and ψ (17) acquires here an additional index i that numbers the current loops (Fig. 2), and the matrix J_i is diagonal in these indices. (The vector indices of the conductances, which were left out of (21), can be easily restored in any specified expression).

Averaging of (18) and (21) over the impurities reduces, when account is taken of (16), to replacing in (13) the action functional $\mathcal{S}$ (14) by the effective functional[4] $S = S_0 + S_{int}$,

$$S_0 = \int \overline{\psi}(\mathbf{r})\left(\frac{1}{2m_e}\nabla^2 + \varepsilon_F + i\eta\Lambda\right)\psi(\mathbf{r})\, d^d r, \tag{22}$$

$$S_{int} = -\frac{i}{2\pi\nu_0\tau}\int \mathrm{Tr}\, q^2\, d^d r; \qquad q(\mathbf{r}) \equiv \psi(\mathbf{r})\otimes\overline{\psi}(\mathbf{r}). \tag{23}$$

The number of replica components in the final expressions is $N = 0$.

In Eq. (23), $q(\mathbf{r})$ is a Hermitian real-quaternion matrix with the structure:

$$q = q^\mu \tau_\mu, \qquad q^\mu = q^{ab;\mu}_{AB;ij}, \qquad (q^\mu)^* = q^\mu. \tag{24}$$

The matrix Λ, which governs the analytic structure of (22) where $\eta \to +0$) indicates the presence of a retarded and of an advanced Green's function in the initial expression (Fig. 2)

$$\Lambda = (\sigma_z)^{ab}\otimes\delta_{AB}\otimes\delta_{ij}\otimes\tau_0. \tag{25}$$

This matrix, just as the prefactor of the exponential in (21), breaks the symmetry of the upper indices. Functional averaging with the action (22) and (23) will be designated by the symbol $\langle\langle\ldots\rangle\rangle_{\chi\varkappa}$.

It is convenient to rewrite expression (21) averaged over the realizations of the random potential by using the generating functional

$$\langle G^n\rangle = \left(\frac{e^2}{16\pi N^2}\right)^n \prod_{i=1}^{n} \mathrm{tr}\frac{\partial^2}{\partial h_i^2}\langle\langle e^{iS[\mathbf{h}]}\rangle\rangle_{\chi\varkappa}\Big|_{h=0}, \tag{26}$$

where the functional $S[\mathbf{h}]$ is proportional to the current matrix (19):

$$S[\mathbf{h}] = \frac{i}{e}\mathrm{Tr}\,\mathbf{h}\mathbf{J}. \tag{27}$$

The structure of the matrix $\mathbf{h}$ is so chosen that the differentiation in (26) does not alter the prefactor in (21):

$$\mathbf{h}^{ab} = \begin{pmatrix} 0 & \mathbf{h}^0_{i;AB} \\ -\mathbf{h}^0_{i;BA} & 0\end{pmatrix}\otimes\tau_0 + \begin{pmatrix} 0 & \mathbf{h}^3_{i;AB} \\ \mathbf{h}^3_{i;BA} & 0\end{pmatrix}\otimes\tau_3.$$

The symbol tr in (26) stands for the trace over all indices, except i and j, which number the current loops, and the notation $\mathbf{h}_{ij} \equiv \mathbf{h}_i\delta_{ij}$ is used.

The next standard step that follows the introduction of

the effective action S is transformation from the "fast" variables $\bar{\psi}$ and ψ with a range of the order of the electron wavelength, to "slow" variables $Q(\mathbf{r})$ that describe the diffusive motion.[4–10,46] This transformation is described for expression (26) in Appendix I, where it is shown that the mean value $\langle\langle ... \rangle\rangle_{\chi\varkappa}$ in (26) is replaced by

$$Z^{-1}\int \mathscr{D}Q \exp\{-F[Q;\mathbf{h}]\}. \tag{28}$$

Here

$$Z=\int \mathscr{D}Q \exp\{-F[Q;\mathbf{h}=0]\},$$

and the generating functional is given, in the principal approximation in l/L, by

$$F[Q;\mathbf{h}]=\frac{1}{t}\int \mathrm{Tr}\left(\nabla Q-\frac{1}{L}[\mathbf{h},Q]\right)^2 d^dr, \tag{29}$$

where $t\equiv \pi\nu D/8$, $D=v_F l/d$ is the diffusion coefficient, and $[\mathbf{h},Q]=\mathbf{h}Q-Q\mathbf{h}$. The matrix $Q(\mathbf{r})$ is Hermitian and has the same structure (24) as the matrix $q(\mathbf{r})$. The nonlinearity of the functional (29) is due to the geometric limits imposed on the field Q:

$$Q^2=I, \quad \mathrm{Tr}\,Q=0. \tag{30}$$

Note that in the presence of an external magnetic field the action (22) would contain a vertex with structure (27), in which $\mathbf{h}$ would be replaced by $\tau_3\mathbf{A}e/c$ ($\mathbf{A}$ is the vector potential of the magnetic field). Therefore, apart from this substitution, the functional (29) is the same as the σ-model functional in a weak external magnetic field.[4]

It is shown in Appendix I that in the derivation of the functional $F[Q;\mathbf{h}]$ there appear, besides (29), vertices of arbitrary power of $(l/L)\mathbf{h}$. The most important, in the sequel, will be vertices with structure

$$\Phi_n[h]=X_n\int \mathrm{Tr}(hQ)^{2n}\frac{d^dr}{L^d}, \quad n>1. \tag{31}$$

The small parameter $(l/L)^{2(n-1)}$ is included in X_n (see Appendix I). It is clear from (26) that the vertices (31) make no contribution to $\langle G\rangle$. In the calculation of $\langle (G)^n\rangle$, on the other hand, account must be taken of all Φ_s with $s\leqslant n$. We shall show that it is precisely the vertices (31) which determine the asymptotic form of the conductance distribution function.

§4. FUNCTIONAL REPRESENTATION OF THE FLUCTUATIONS OF THE DENSITY OF STATES

The density of states in any realization of a random potential is written as usual in terms of the exact Green's functions as

$$\nu(\varepsilon)=\frac{i}{\pi L^d}\int [\mathscr{G}^r(\varepsilon;\mathbf{r},\mathbf{r})-\mathscr{G}^a(\varepsilon;\mathbf{r},\mathbf{r})]d^dr. \tag{32}$$

Using the functional averaging (13) introduced in §3, we rewrite this expression in the form

$$\nu(\varepsilon)=-\frac{1}{\pi L^d}\left\langle\int \chi^1(\mathbf{r})\varkappa^1(\mathbf{r})-\chi^2(\mathbf{r})\varkappa^2(\mathbf{r})]d^dr\right\rangle_{\chi\varkappa}, \tag{33}$$

which reduces, after transforming to the fields $\bar{\psi}$ and ψ (17), to

$$\nu(\varepsilon)=-\frac{1}{\pi NL^d}\left\langle\int [\bar{\psi}(\mathbf{r})\Lambda\psi(\mathbf{r})]d^dr\right\rangle_{\chi\varkappa}. \tag{33a}$$

We are interested in the nth power of the density of states for a given energy ε. It is convenient to express this quantity in analogy with (26) with the aid of a generating functional. Averaging, as in §3, over the realizations of the random potential, we get

$$\langle \nu^n(\varepsilon)\rangle=\left(\frac{\nu_0}{N}\right)^n\prod_{i=1}^n\frac{\partial}{\partial\omega_i}\langle\langle e^{iS[\omega]}\rangle\rangle_{\chi\varkappa}\Big|_{\substack{N=0\\ \omega=0}}, \tag{34}$$

where

$$S[\omega]=-\frac{t}{\pi\nu_0 L^d}\int \mathrm{Tr}(\omega\Lambda q)d^dr. \tag{35}$$

Here $q(\mathbf{r})$ is the matrix field (23), ω is a matrix having the index structure of (24) and the form

$$\omega=\omega_i\delta_{ij}\otimes\delta^{ab}\otimes\delta_{AB}\otimes\tau_0, \tag{36}$$

Λ is the matrix defined in (25), and $\langle\nu\rangle=2\nu_0$ is the average density of states and is independent of the sample size L.

Unlike (34), the quantity investigated in Ref. 27 was the state-density correlator $\langle\nu(\varepsilon_1)\nu(\varepsilon_2)\rangle$ at different energies. The point is that for the isolated sample considered in Ref. 27 we have $\langle\nu(\varepsilon_1)\nu(\varepsilon_2)\rangle\to\infty$ as $|\varepsilon_1-\varepsilon_2|\to 0$. Here we discuss a sample making contacts with a bulky metal. For such a sample $\langle\nu^2(\varepsilon)\rangle$ is not singular, owing to the broadening of the energy levels. If the level broadening in an isolated sample is $\gamma\gtrsim D/L^2$, the results that follow hold also for this case with γ replacing D/L^2.

As a result of the change to the slow variables $Q(\mathbf{r})$, the mean value $\langle\langle ... \rangle\rangle_{\chi\varkappa}$ of (34) is transformed, as above, into (28), where the functional (29) acquires, in the leading approximation in l/L, an additional vertex

$$F[\omega]=-\frac{1}{2}\int \mathrm{Tr}(\omega\Lambda Q)\frac{d^dr}{L^d}. \tag{37}$$

In the calculation of the fluctuation moments of the density of states (34) it is necessary to take into account also vertices containing higher powers of $\omega\Lambda$:

$$\Phi_m[\omega]=Y_m\int \mathrm{Tr}(\omega\Lambda Q)^m\frac{d^dr}{L^d}. \tag{38}$$

An equation for coefficients Y_m that are small in the parameter $(l/L)^m$ is given in Appendix I.

Expression (35) and hence (37) is similar in structure to that entering in the σ-model functional used for the analysis of the conductivity at a finite frequency ω (Refs. 3–5). The essential difference is that in our case ω is the matrix (36). Vertices with high powers of $\omega\Lambda$, including those of type (38), were first considered in Ref. 10 in an examination of the instability of one-parameter scaling in the Anderson localization problem.

We can write also a general functional expression for mixed fluctuation moments of the conductance and of the density of states:

$$\langle G^n\nu^m\rangle=\left(\frac{e^2}{16\pi N^2}\right)^n\left(\frac{\nu_0}{N}\right)^m\prod_{i\leqslant n<j\leqslant n+m}\left(\frac{\partial}{\partial\omega_i}\right)\times\left(\mathrm{tr}\frac{\partial^2}{\partial h_j^2}\right)\int\frac{\mathscr{D}Q}{Z}e^{-F[Q;\mathbf{h};\omega]}\Big|_{(N,\mathbf{h},\omega)=0}. \tag{39}$$

Here

$$F[Q;h;\omega]=\int \mathrm{Tr}\left\{\frac{1}{t}\left(\nabla Q-\frac{1}{L}[\mathbf{h},Q]\right)^2-\frac{1}{2L^d}\omega\Lambda Q\right\}d^dr \tag{40}$$

$$+\sum_{n+m>1}X_{nm}(\mathscr{P})\int \mathrm{Tr}\,\mathscr{P}\{(hQ)^{2n}(\omega\Lambda Q)^m\}\frac{d^dr}{L^d}. \tag{41}$$

The summation in (42) is over all permutations $\mathscr{P}$ of the noncommuting matrices hQ and $\omega\Lambda Q$. The permutation-dependent coefficient $X_{nm}(\mathscr{P})$ is proportional to $(l/L)^{2(n+m-1)}$. We shall hereafter be interested in the cumulants, i.e., those parts of the fluctuation moments which are determined only by connective perturbation-theory diagrams.

§5. PERTURBATION THEORY

The formalism developed in §§3, 4 permits a perturbation theory to be constructed in regular fashion. For practical calculations it is necessary to change from the variables $Q(\mathbf{r})$ to independent variables in such a way that the constraints (30) are automatically satisfied. It is most convenient to parametrize the field Q in a manner similar to that introduced in the supersymmetric approach of Ref. 9:

$$Q=(1-W/2)\Lambda(1-W/2)^{-1}. \tag{42}$$

Here $W(\mathbf{r})$ is an anti-Hermitian matrix of form (24), antidiagonal in the superior indices a and b, and arbitrary in all other indices. The functional integration in (28) is now over the independent variables $W^{12}(\mathbf{r})=-[W^{21}(\mathbf{r})]^+$. The advantage of the parametrization (42) is that in the re
limit $N=0$ the Jacobian of the transition from $Q(\mathbf{r})$ to W is unity.

The perturbation theory is constructed in terms of dimensionless parameter

$$g_0^{-1}=\frac{e^2}{\hbar G_0}\alpha_d\equiv\frac{t_0 l^{2-d}}{16\pi}\alpha_d\ll 1; \quad \alpha_d=\frac{16\Gamma(2-d/2)}{(4\pi)^{d/2+1}}.$$

The functional of the zeroth approximation is F_0, obtain from $F\equiv F[Q;h=0;\omega=0]$ (40) in lowest order of the pansion in W:

$$F_0=-\frac{1}{t}\int \mathrm{Tr}(\nabla W)^2 d^dr=\frac{2}{t}\int \mathrm{Tr}(\nabla W^{12}(\nabla W^{12})^+)dr^d.$$

It is expedient to calculate in (39) first the functional int gral, and then take the derivatives with respect to the sourc $\mathbf{h}$ and ω. In this case $\exp\{-F[Q;\mathbf{h};\omega]\}$ is expanded powers of $F-F_0$ and is averaged with a weight $\exp(-F_0$

$$\langle ...\rangle_0=\int \mathscr{D}W^{12}(...)e^{-F_0}\Big/\int \mathscr{D}W^{12}e^{-F_0}.$$

The Gaussian mean values are calculated in the momentum representation by using an equation that follows directly from (44) and (45):

$$\langle W^{12;\mu}_{AB;ij}(\mathbf{q})W^{12;\nu}_{CD;kl}(\mathbf{q}')\rangle_0=\frac{t}{8q^2}\delta(\mathbf{q}+\mathbf{q}')\delta_{Ac}\delta_{BD}\delta_{ik}\delta_{jl}\delta^{\mu\nu}. \tag{46}$$

The subsequent calculations are considerably simplified by using the following identities[4)] which can be verified by using (46):

$$\langle \mathrm{Tr}(W(\mathbf{q})PW(\mathbf{q}')R)\rangle_0=\frac{t}{8q^2}\delta(\mathbf{q}+\mathbf{q}')\{\mathrm{Tr}(\Lambda P\Lambda R^+-PR^+)+\mathrm{Tr}\,\Lambda P\,\mathrm{Tr}\,\Lambda R-\mathrm{Tr}\,P\,\mathrm{Tr}\,R\}, \tag{47}$$

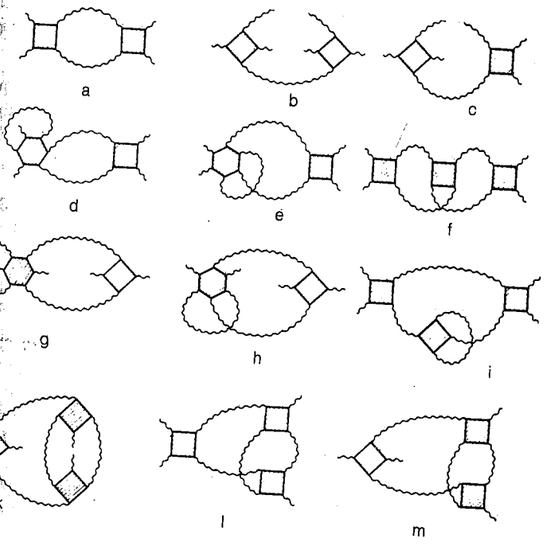

FIG. 3. Diagrams for the conductance variance: squares with two and one free vertices represent the first terms of the expansions of $\mathrm{Tr}[\mathbf{h},W]^2$ and $\mathrm{Tr}\mathbf{h}Q\nabla Q$ in terms of W, respectively. A square without free vertices corresponds to $\mathrm{Tr}(\nabla W)^2W^2$ which appears upon expansion of $\exp(-F)$ in (40) ($F\equiv F[Q;\mathbf{h}=0,\omega=0]$).

$$\langle \mathrm{Tr}(W(\mathbf{q})P)\,\mathrm{Tr}(W(\mathbf{q}')R)\rangle_0$$

$$= \frac{t}{8q^2}\delta(\mathbf{q}+\mathbf{q}')\,\mathrm{Tr}(\Lambda P\Lambda R - \Lambda P\Lambda R^+ - PR + PR^+). \qquad (48)$$

Here P and R are arbitrary real quaterion matrices and Λ is the matrix (25). Functional mean values containing more than two matrices W, which appear when Q is expanded in terms of W in accordance with (42), are calculated by using the Wick theorem and Eqs. (47) and (48).

We illustrate first the developed formalism with the calculation of the mean squared conductance fluctuations $\langle(\delta G)^2\rangle$ in the leading approximation in l/L as the example. We use for this purpose Eqs. (39) and (40) with $n=2$ and $m=0$.

A contribution to the cumulant $\langle(\delta G)^2\rangle$ in the lowest (second) order of perturbation theory is made only by the vertex $\mathrm{Tr}[\mathbf{h}, Q]^2$. This contribution, obtained by expanding in accordance with (42) each of such vertices up to second order in W, is best illustrated using the diagrams of Figs. 3a and 3b. In any order of expansion in W, the quantity $\mathrm{Tr}[\mathbf{h}, Q]^2$ is represented by a polygon with two free vertices corresponding to $\mathbf{h}$ and the remaining vertices to W; wavy lines denote the pairings $\langle WW\rangle_0$, i.e., diffusons (or cooperons)

$$t/q^2 \propto (\pi\nu_0)^{-1}(Dq^2 - i\eta)^{-1}$$

[see (47) and (48)]. The analogy between these diagrams and those of the impurity ("crossover") technique[2] is obvious. Note that the additional dashed lines that appear in the polygons of the impurity technique[2] are taken into account here automatically, just as in the σ-model formalism for the perturbation-theory calculation of the density-density correlator.[4,7]

Diagram 3c, which is impossible in the impurity technique because it does not correspond to two current loops joined by a dashed line, turns out in the present approach to be proportional to $\mathrm{Tr}h^4$. Differentiation of (39) causes this contribution to vanish, since $\mathrm{Tr}h^4$ contains no cross $(i\neq j)$ terms because the source $\mathbf{h}$ is diagonal in the indices i and j that number different current loops, $\mathbf{h}_{ij} \equiv \mathbf{h}_i\delta_{ij}$. In the general case of calculation of an arbitrary fluctuation moment the differentiation with respect to the sources causes vanishing of all contributions except those that have the structure $(\mathrm{Tr}h^2)^n\ (\mathrm{Tr}\omega)^m$ after (47) (48) are averaged.

Diagrams 3a and 3b are proportional respectively to $(\mathrm{Tr}h^2)^2$ and $(\mathrm{Tr}h_\alpha h_\beta)^2$. Differentiating (39) using the relation

$$\mathrm{tr}\frac{\partial^2}{\partial h_{\alpha,i}\partial h_{\beta,j}}[\mathrm{Tr}\,h_\gamma h_\delta] = 32N^2(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}) \qquad (49)$$

we obtain for these diagrams the result

$$\langle \delta G_{\alpha\beta}\delta G_{\gamma\delta}\rangle = 4\left(\frac{e^2}{\pi\hbar}\right)^2 L^{d-4}I_4(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}) \qquad (50)$$

which coincides with the result[24-27] of the impurity diagram technique. Here

$$I_s = \int \frac{d^dq}{(2\pi)^d}\frac{1}{q^s}, \qquad (51)$$

FIG. 4. Diagram for the variance of the density of states in the lowest order in g_0^{-1}. A triangle represents the first term of the expansion of the vertex $\mathrm{Tr}\Lambda Q$ in terms of W.

where the infrared divergence is cut off by the reciprocal of the system dimension.

The succeeding perturbation-theory terms introduce in (50) corrections of relative value proportional to $(g_0^{-1}I_2)^n$. At $d\leqslant 2$ these corrections are not small if the sample size L is large enough, and they must be summed. The corresponding diagrams of order $g_0^{-1}I_2I_4L^{d-4}$, generated by the vertex $\mathrm{Tr}[\mathbf{h}, Q]^2$, are shown in Figs. 3d–3h.[5] Direct calculation using (47) and (48) shows that diagrams 3d,e,f and, independently, 3g,h are mutually cancelled. Also cancelled are the analogous corrections to (50), not shown in this figure, and due to the vertex $\mathrm{Tr}\mathbf{h}Q\nabla Q$.

It will be shown below that the corrections to $\langle(\delta Q)^2\rangle$, which are proportional to $(g_0^{-1}I_2)^n$ (the principal logarithmic corrections if $d=2$) cancel out in all orders of perturbation theory.

Figures 3k,l,m show the nonlogarithmic corrections to (50); their relative value is $\sim g_0^{-1}$. These corrections do not cancel out. Of course, they are negligibly small compared with (50). It will be shown below, however, that the logarithmic encumbrances in these diagrams make the variance of the conductance non-universal in the dielectric region, i.e., at $L\sim L_c$, where $L_c \sim l\exp g_0$ is the localization radius at $d=2$. The mean squared fluctuations of the density of states are obtained from (39) and (40) at $n=0$ and $m=2$. In the lowest order of perturbation theory they correspond to a single diagram (Fig. 4) that coincides, naturally, with the diagram[27] of the usual impurity technique. As a result,

$$\langle(\delta\nu)^2\rangle = \left(\frac{4e^2\nu_0}{\pi\hbar G_0}\right)^2 L^{d-4}I_4\,. \qquad (52)$$

The corrections to $\langle(\delta\nu)^2\rangle$ which are not proportional to $(g_0^{-1}I_2)^n$ are not cancelled. In the next section we shall sum them by the renormalization-group method and also prove that there are no renormalizations in $\langle(\delta G)^2\rangle$.

§6. EXACT RELATIONS AND ONE-PARAMETER SCALING

An effective method of summing logarithmic corrections is known to be the renormalization of all the charges on which the functional depends, followed by calculation of the required quantities (e.g., cumulants), in first nonvanishing order of perturbation theory, with the aid of a renormalized functional. We consider in §6 only the renormalization of the functional (40) that yields both the results (50) and (52) for the second-order cumulants, and their logarithmic corrections that contain no powers of the small parameter L.

We rewrite this functional in a form that shows explicitly its dependence on all the renormalization-group charges

the number of which is formally equal to the number of vertices:

$$F[Q;\mathbf{h};\omega] = \int \mathrm{Tr}\left\{\frac{1}{t}(\nabla Q)^2 - \frac{4z_1}{tL}\mathbf{h}Q\nabla Q + \frac{z_2}{tL^2}[\mathbf{h}, Q]^2 - \frac{z_3}{2L^d}\omega\Lambda Q\right\}d^dr. \qquad (53)$$

We shall prove that in the functional (53) the renormalization-group charges z_{1-3}, whose unrenormalized values are equal to unity, are not renormalized. This means that the functional (40) depends in fact on a single charge $t\propto g^{-1}$, i.e., on the dimensionless conductance, and that the one-parameter-scaling hypothesis is valid for this functional.

No calculations whatever are needed to prove that the charges z are not renormalized. The fact that the charge z_3 is not renormalized follows from the already noted coincidence of the vertex (37) with the vertex that describes the frequency dependence of the conductivity. The charge at this vertex is not renormalized[3-5] by virtue of the particle-number conservation law. Note that the identity $z_3=1$ leads, in particular, to the known statement that there are no logarithmic corrections to the density of states, since $\langle\nu\rangle$ is obtained after differentiating the functional (53) with respect to ω [see (39)].

The average conductance $\langle G\rangle$ is determined in the lowest order of perturbation theory, as follows from (39) and (40), by the vertex $\mathrm{Tr}[\mathbf{h}, Q]^2$. The renormalized, i.e., observed, value $\langle G\rangle$ is proportional to the renormalized charge z_2t^{-1} at this vertex.[46] On the other hand the charge t^{-1} at the $\mathrm{Tr}(\nabla Q)^2$ vertex determines the renormalized value of the diffusion coefficient. Consequently,

$$\langle G\rangle = \frac{16z_2}{t}\frac{e^2}{\pi\hbar}L^{d-2} = z_2(2e^2\nu_0D)L^{d-2}. \qquad (54)$$

It is clear therefore that the condition $z_2=1$ which is valid in the unrenormalized case is preserved also upon renormalization, inasmuch as only under this condition does (54) go over into the Einstein relation which, of course, should remain in force. This conclusion is confirmed also by a direct calculation in the one-loop approximation.[46]

The functional (40) can be used also to calculate the local conductivity

$$\sigma_{\alpha\beta}(\mathbf{r}-\mathbf{r}') = \langle\delta j_\alpha(\mathbf{r})/\delta E_\beta(\mathbf{r}')\rangle.$$

To this end, the source $\mathbf{h}$ should be regarded as a field that depends on $\mathbf{r}$, and the partial derivative $\partial^2/\partial h_\alpha\partial h_\beta$ in (39) must be replaced by the functional one $\partial^2/\partial h_\alpha(\mathbf{r})\delta h_\beta(\mathbf{r}')$. After this replacement we obtain from (39) and (53), in lowest-order perturbation theory,

$$\sigma_{\alpha\beta}(\mathbf{q}) = 2e^2\nu_0D(z_2\delta_{\alpha\beta} - z_1^2q_\alpha q_\beta/q^2). \qquad (55)$$

In the static limit considered here it follows from the continuity equation $\mathrm{div}\,\mathbf{j}=0$ that $\sigma_{\alpha\beta}(\mathbf{q})$ should have the structure of a transverse projector. It is clear from (55) that such a structure is produced only if $z_1^2=z_2$. Since the continuity equation cannot be violated in renormalizations, the condition $z_2=1$ leads also to the condition $z_1=1$.

Thus, the particle-number conservation law, the Einstein relation, and the continuity equation result in a one-parameter renormalization group for the functional (53). This means that summation of all the corrections containing $g_0^{-n}(I_2)^m$ (at $d=2$ we have $I_2\sim\ln L/l$) reduces to replacing the classical value of the conductance $g_0(L/l)^{d-2}$ in all the expression by its exact (renormalized) value g (at $d=2$ and in the one-loop approximation we have $g=g_0-\ln L/l$).

Since the mean squared conductance fluctuations (50) are independent of g_0, the principal logarithmic corrections $(g_0^{-1}I_2)^2$ to this quantity cancel out. There remain, however, corrections of order $g_0^{-n-1}I_2^n$, obtained from the more complicated diagrams 3k, 3l, and 3m which are proportional to g_0^{-1}. Summation of these corrections adds to (50) a correction $\sim g^{-1}$ that becomes substantial at scales exceeding the localization length L_c. In these scales, the dispersion of the conductance is no longer a universal quantity. In the metallic region $g>1$, on the other hand, the dispersion $\langle(\delta G)^2\rangle$ remains constant (Fig. 1a). The corrections of the mean squared density fluctuations are accounted for by replacing G_0 by G in this expression. As a result we arrive at the plots shown in Fig. 1 for the mean squared fluctuations as functions of scale.

We emphasize that the exact relations considered here impose no contraints on the charge renormalization in the additional generating functional (41). The point is that these relations were obtained by considering mean values. The functional (41), however, does not contribute to the mean values, since it contains higher powers of the sources ω and $\mathbf{h}$.

§7. HIGHER-ORDER CUMULANTS. NON-GAUSSIAN DISTRIBUTION FUNCTION

We consider in this section any "normal" contributions to the higher-order cumulants, as obtained from the generating functional (40) without allowance for (41). In accordance with the results of the preceding section, only nonlogarithmic perturbation-theory contributions need to be taken into account in the calculation of the cumulants, using the nonrenormalized value of g. We confine ourselves to the lowest order of perturbation theory and disregard the contributions $\sim g^{-n}$, which are small in the metallic region.

A diagrammatic representation for the simplest non-Gaussian cumulant $\langle(\delta\nu)^3\rangle$ is shown in Fig. 5. Analytic expressions for all the contributions are obtained from (39) and (40) with $n=0$ and $m=3$. The diagram 5c is proportional to $I_2I_6g_0^{-4}L^{-2d}$, were I_s is determined by (51), i.e., at $d=2$ this diagram is proportional to $g_0^{-4}\ln L/l$. Such corrections must cancel out in the renormalizable one-parameter theory. The point is that all the logarithmic corrections arise in perturbation theory on account of the expansion of the single charge $g^{-1}=(g_0-\ln L/l)^{-1}$. Therefore to any diagram containing $\ln L/l$ there must correspond a nonlogarithmic diagram of lower order in g_0^{-1}. On the other hand, there exist no diagrams proportional to $g_0^{-3}I_6L^{-2d}$.

Indeed, direct calculation shows that diagram 5c is cancelled by the corresponding contribution of diagram 5d. The latter contains, besides $I_2I_6g_0^{-4}L^{-2d}$, a contirbution pro-

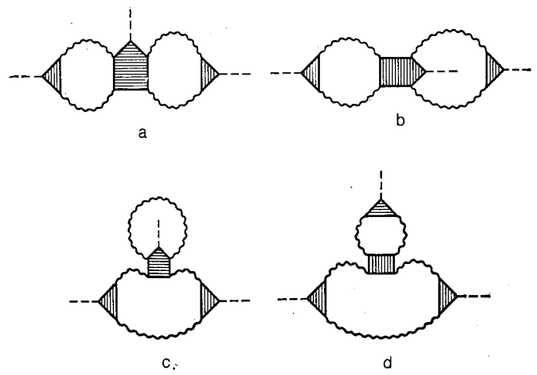

FIG. 5. Diagrams for $\langle(\delta\nu)^3\rangle$.

portional to $(I_4)^2 g_0^{-4} L^{-2d}$, which cancels out the contributions of diagrams 5a and 5b. It is thus found that $\langle(\delta\nu)^3\rangle = 0$ in the leading approximation in l/L.

The cancellations of all the contributions to $\langle(\delta\nu)^3\rangle$ is accidental, since it is due neither to the renormalizability requirement nor to any other general relation. The remaining third-order cumulants differ from zero. We present the results for the mixed moment $\langle(\delta\nu)^2\delta G\rangle$.

Using (39) and (40) with $n=1$ and $m=2$, as well as the diagrammatic formalism developed in §5, we obtain after replacing $g_0(L/l)^{d-2}$ by g:

$$\frac{\langle(\delta\nu)^2\delta G_{\alpha\beta}\rangle_c}{\langle\nu\rangle^2\langle G\rangle} = \delta_{\alpha\beta}\left(\frac{2\pi}{g}\right)^4 L^{2d-8}\left[(I_4)^2 - \frac{2}{d}(I_4^2+I)\right], \quad (56)$$

where

$$I = 4\int \frac{d^dq\, d^dq'}{(2\pi)^{2d}} \frac{1}{q^4 q'^2 (\mathbf{q}+\mathbf{q}')^2}.$$

The contribution proportional to $1/d$ in (56) is determined by the vertex $\mathrm{Tr}\,\mathbf{h}Q\nabla Q$. Note that this vertex made no contribution in the lower-order perturbation theory to either $\langle G\rangle$ or $\langle(\delta G)^2\rangle$.

The relative value of the cumulants $\langle\delta\nu(\delta G)^2\rangle_c$ and $\langle(\delta G)^3\rangle_c$ is also found too be of the order of (56). Comparing (50) with (56) we find that in the metallic conductivity region $g \gg 1$ the non-Gaussian corrections are small. Near the Anderson transition ($g \lesssim 1$), however, the third-order cumulants become comparable with the second-order ones.

Let us estimate the order of magnitude of the contribution made to the higher cumulants by the functional (40). The cumulant $\langle(\delta G)^n(\delta\nu)^m\rangle_c$, is a sum of diagrams containing $n+m-1$ loops and having m vertices $\mathrm{Tr}\,\omega\Lambda Q$, k vertices $\mathrm{Tr}(\mathbf{h}Q)^2$, and $2(n-k)$ vertices $\mathrm{Tr}\mathbf{h}Q\nabla Q$ ($0 \leqslant k \leqslant n$). The contributions containing I_s with $s \geqslant 6$ must contain also the integrals I_2, which are logarithmic at $d=2$. These contributions, as already mentioned, cancel out by virtue of the renormalizability of the theory. Therefore each of the $n+m-1$ independent integrations over the momentum yields a factor of order $g^{-2}L^{4-d}$; each vertex $\mathrm{Tr}(\mathbf{h}Q)^2$ (just as a pair of $\mathrm{Tr}\,\mathbf{h}Q\nabla Q$ vertices) leads to a factor g. It is clear thus that the contribution of each such diagram is proportional to $g^{-n-2m+2}$, which goes over into (6) at $m=0$.

§8. ADDITIONAL CONTRIBUTIONS TO CUMULANTS

The functional (40) is insufficient for the calculation of the higher-order moments, and hence also of the distribution function. We shall show that these moments are determined by the contribution of the functional (41), notwithstanding the small unrenormalized value $(l/L)^{2(n+m-1)}$ of the charges it contains. (In the present section we consider only the two-dimensional case.)

We identify first the perturbation-theory diagrams to which allowance for the functional (41) corresponds when the cumulant $\langle G^n\rangle_c$ is calculated. A contribution to (39) is made by the vertex $\mathrm{Tr}(hQ)^{2n}$ if it breaks up, after functional integration with respect to Q, into a product of matrix traces $(\mathrm{Tr}h^2)^n$. It follows from the averaging formula (47) that this calls for at least $n-1$ W-pairings. In accordance with the rules of §5, the quantity $\mathrm{Tr}(hQ)^{2n}$ averaged in this manner is represented by a $(4n-2)$-sided polygon having $2n$ free vertices and $2n-2$ vertices pairwise joined by wavy lines (Fig. 6). The contribution of such a diagram to the cumulant $\langle G^n\rangle_c^{\mathrm{add}}$ is proportional to $(l/L)^{2n}(g_0^{-1}\ln L/l)^{n-1}$, i.e., is small compared with the contribution (6) of the diagrams that stem from the functional (40). However, the number of these "additional" diagrams is proportional to n^2. This is easily demonstrated by considering the corresponding diagram of the usual crossover technique (Fig. 6). The two boundary current loops are distinguished from among the total number n, meaning that the number of possible diffusion arrangements is of the order of $C_n^2 \sim n^2$.

To obtain the correction to the value of $\langle G^n\rangle_c^{\mathrm{add}}$ in the next order in $g_0^{-1}\ln L/l$, one more diffuson must be added. In each of the diagrams of Fig. 6 this can be done in $\approx n^2$ ways, so that n^3 equivalent correction diagrams are produced. In each succeding perturbation-theory order the number of diagrams increases by approximately n^2 times. Thus, the additional contribution to the cumulant, $\langle G^n\rangle_c^{\mathrm{add}}$ contains a series in terms of the parameter $n^2 g_0^{-1}\ln L/l$.

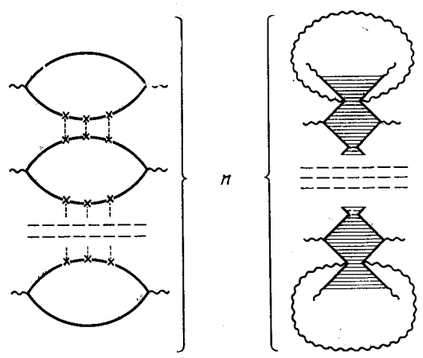

FIG. 6. Additional contribution to the nth cumulant in the lowest order of perturbation theory.

We shall show that summation of this series, which is essential because of the topologically large number of diagrams, compensates for the unrenormalized smallness of $\langle G^n\rangle_c^{\mathrm{add}}$. An effective method of summing these diagrams, which can of course not be implemented directly, is a renormalization-group analysis of the generating functional (41).

We consider now the vertex (31) of the functional (41), which makes a contribution $\langle G^n\rangle_c^{\mathrm{add}}$ to the nth moment of the conductance fluctuations. It is shown in Appendix 2 that renormalization of this vertex generates in succession vertices continuing $2n$ matrices h and increases continuously the number of matrix traces. The most important at $n \gg 1$ are the vertices in which the number of fields Q is equal to its maximum $2n$:

$$\Phi_n^{\mathbf{s}}[h;Q] = X_n^{\mathbf{s}} \int \{[\mathrm{Tr}(hQ)^2]^{s_1}\ldots[\mathrm{Tr}(hQ)^{2m}]^{s_m}\ldots\}\frac{d^dr}{L^d} \equiv X_n^{\mathbf{s}}|\mathbf{s}\rangle, \quad (57)$$

where $\mathbf{s} = (s_1,\ldots,s_m,\ldots)$ is a set of natural numbers satisfying the condition

$$\sum_{m\geqslant 1} m s_m = n. \quad (58)$$

It is necessary to include in the renormalization-group scheme all the possible vertices $\Phi_n^{\mathbf{s}}$ (57).

The normalization-group equations for the charges $X_n^{\mathbf{s}}$ are derived in the one-loop approximation in Appendix 2 and can be symbolically written in the form

$$\sigma\frac{dX_n^{\mathbf{s}}}{d\ln L/l} \equiv \frac{dX_n^{\mathbf{s}}}{du} = \sum_{(\mathbf{s}')} L_n^{\mathbf{s}\mathbf{s}'} X_n^{\mathbf{s}'}, \quad (59)$$

where

$$u = \ln(g_0/g), \quad (60)$$

and the first equality of (59) follows directly from (1) and (2). In Eq. (59), $L_n^{\mathbf{s}\mathbf{s}'}$ is a matrix that is independent of u and whose rank is equal to the number of different sets $\{\mathbf{s}\}$ (58), i.e. to the number of ways that n can be broken up into a sum of natural numbers.

The matrix equation (59) can be represented in an operator form that follows directly from Eq. (A.II.7) (see Appendix 2):

$$-\frac{d\Psi_n}{du} = \hat{L}\Psi_n, \qquad \Psi_n = \sum_{(\mathbf{s})} X_n^{\mathbf{s}}|\mathbf{s}\rangle, \quad (61)$$

where

$$\hat{L} = \sum_{m\geqslant 1} m a_m^+ a_m + \sum_{m,l\geqslant 1}\{(m+l)a_m^+ a_l^+ a_{m+l} + 4ml a_{m+l}^+ a_m a_l\}, \quad (62)$$

and the "creation" and "annihilation" operators are

$$a_m^+|s_1,\ldots,s_m,\ldots\rangle = |s_1,\ldots,s_m+1,\ldots\rangle,$$
$$a_m|s_1,\ldots,s_m,\ldots\rangle = s_m|s_1,\ldots,s_m-1,\ldots\rangle. \quad (63)$$

The initial condition for Eqs. (61) at $u=0$ is of the form

$$\Psi_n(u=0) = X_n^{(0,\ldots,0,1)}|0,\ldots,0,1\rangle, \quad (64)$$

since the unrenormalized generating functional (41) contains (at $\omega=0$) only the vertices (31). At large n, the value of $X_n(0)$ is proportional to $g_0(2l/L)^{2n-2}n^{-2}$ (the exact values of the unrenormalized charges $X_n(0)$ are given in Appendiix I, see (A.I.8)). At arbitrary u, the solution of (61) takes the form

$$\Psi_n(u) = \sum_i c_i \Psi_n^{(i)} e^{-E_n^{(i)}u} = \sum_i c_i \Psi_n^{(i)}\left(\frac{g_0}{g}\right)^{-E_n^{(i)}}, \quad (65)$$

where $\Psi_n^{(i)}$ is an eigenvector of the operator (62), $E_n^{(i)}$ is the corresponding eigenvalue, and c_i are the coefficients of the eigenvector expansion of the initial state (64).

The asymptotics of $\Psi_n(u)$, i.e., the asymptotics of the renormalization-group charges $X_n^s(u)$ for sufficiently large u, are determined by the smallest eigenvalue $E_n^{(0)}$. It is remarkable that the eigenvector $\Psi_n^{(0)}$ corresponding to the smallest eigenvalue $E_n^{(0)}$ can be written in the explicit form:[6)]

$$\Psi_n^{(0)} = \sum_{\{s\}}\left[\prod_{m\geqslant 1}(-2m)^{s_m} s_m!\right]^{-1}|s_1,\ldots,s_m,\ldots\rangle. \quad (66)$$

Acting on this vector by the operator (62), we can directly verify that

$$\hat{L}\Psi_n^{(0)} = E_n^{(0)}\Psi_n^{(0)}, \qquad E_n^{(0)} = -(2n^2-3n). \quad (67)$$

Thus, the growth of all the charges X_n^s in (57) following renormalization-group transformations is given by

$$X_n^s \propto g_0\left(\frac{l}{L}\right)^{2(n-1)}\left(\frac{g_0}{g}\right)^{2n^2-3n}, \quad (68)$$

where g_0 is the unrenormalized and g the normalized [in accordance with (3)] value of the dimensionless conductance. The contribution of the functional (41) to the nth moment of the conductance is proportional, according to (39), to the charge $X_n^{(n,0,\ldots,0,0)}$. By calculating the functional derivative of (39), we obtain the additional contribution (9) to the fluctuation moment $\langle G^n\rangle_c$.

A similar growth takes place also for the additional contribution to the nth fluctuation moment of the density of states, defined by the vertices

$$\Phi_n^{\mathbf{s}}[\omega;Q] = Y_n^{\mathbf{s}}\int\{[\mathrm{Tr}(\omega\Lambda Q)]^{s_1}\ldots[\mathrm{Tr}(\omega\Lambda Q)^m]^{s_m}\ldots\}\frac{d^dr}{L^d} \quad (69)$$

with $\{\mathbf{s}\}$ satisfying the condition (58). These vertices are generated in succession upon renormalization of the vertex (38) that enters in the unrenormalized generating functional (41). The change of these vertices in renormalization-group transformations is also described by Eqs. (61) and (65), where the operator $\hat{L}_\omega$ derived in a somewhat different notation in Ref. 10, differs slightly in form from (62):

$$\hat{L}_\omega = -\sum_{m\geqslant 1}\frac{m(m-1)}{2}a_m^+ a_m + \sum_{m,l\geqslant 1}\left\{\frac{m+l}{2}a_m^+ a_l^+ a_{m+l} + ml a_{m+l}^+ a_m a_l\right\}. \quad (70)$$

The smallest eigenvalue corresponding to this operator is found to be

$$E_n^{(0)} = -(n^2-n). \tag{71}$$

The rules (71) and (65) were asserted in Ref. 10 for small n and proved asymptotically for $n \gg 1$. That they are exact for arbitrary n can be verifying by applying the operator (70) to the same eigenvector (66) as in the case of conductance fluctuations. In the upshot we arrive at a variation of Y_n^s similar to that in (68) for X_n^s. Using (39), we find the contribution of the functional (41) to the nth moment of the density of states:

$$\langle \nu^n \rangle_c^{add} = \frac{\nu_0}{n-1/2} \left(\frac{4\pi\nu_0 l^2}{g_0 L^2} \right)^{n-1} \left(\frac{g_0}{g} \right)^{n^2-n}. \tag{72}$$

Note that in contrast to expression (9) for $\langle G^n \rangle_c^{add}$, expression (72) for the additional contribution to the fluctuation moment of the density of states it is possible to determine even the numerical coefficient.

Substituting in (72) the expression $l/L = \exp(g - g_0)$ that follows from (1) and (2), we find that this contribution becomes large for the momentum with number

$$n \gtrsim 2\Delta^2 \equiv \frac{2(g_0-g)}{\ln g_0/g} = \begin{cases} 2g_0, & g \gg 1 \\ 2g_0/\ln g_0, & g \sim 1 \end{cases}. \tag{73}$$

For the conductance fluctuations, the additional contribution (9) becomes significant at $n \gtrsim \Delta^2$. In particular, in the weak-localization region ($g \gg \ln L/l$) we get the condition $n \gtrsim g_0$.

Let us estimate the contribution to $\langle \nu^n \rangle_c^{add}$ and $\langle G^n \rangle_c^{add}$ from the eigenvalues that follow the smallest ones. It follows from (65) that this contribution is small in the parameter $\exp[(E_n^{(0)} - E_n^{(1)})u]$. The eigenvalue that follows the smallest one turns out to be $-(2n^2-7n)$ for the operator (62) and $-(n^2-3n+1)$ for the operator (70). We have therefore under the condition (73)

$$e^{(E_0^{(1)} - E_n^{(1)})u} < e^{-4(g_0-g)} = \left(\frac{l}{L} \right)^4 \ll 1 \tag{74}$$

which follows from (60), (67), and (71), so that the contribution of the eigenvalues that follow $E_n^{(0)}$ can be neglected for all u.

§9. ASYMPTOTIC FORMS OF DISTRIBUTION FUNCTIONS

Knowing the cumulants, we can reconstruct the distribution functions $f(x)$ of the conductance fluctuations ($x = \delta g$) of the density of states ($x = \delta\nu$) by using the known formula

$$f(x) = \int_{-\infty}^{\infty} e^{i\xi x} \exp\left\{ \sum_{n \geq 2} K_n \frac{(-i\xi)^n}{n!} \right\} \frac{d\xi}{2\pi} = \int_{-\infty}^{\infty} e^{i\xi x} f(\xi) \frac{d\xi}{2\pi}. \tag{75}$$

The cumulant K_n is the sum of the "normal" contribution (see §7) and of the additional contribution (9) or (72). The distribution function can therefore be written in the form of a convolution:

$$f(x) = \int_{-\infty}^{\infty} f^N(x-y) f^{add}(y)\, dy, \tag{76}$$

where $f^N(d)$ and $f^{add}(x)$ are distribution functions that lead respectively only to "normal" or only to "additional" contributions to (75). The function $f^N(x)$ is close to Gaussian, since the normal non-Gaussian cumulants are small in the weak-localization region $g \gg 1$ (see §7).

To calculate $f^{add}(x)$, we note that with increase of n the cumulant K_n^{add} increases so rapidly that for values of n satisfying (73) we have

$$K_n \gg C_n^m K_m K_{n-m}. \tag{77}$$

To calculate $f^{add}(x)$ we can therefore leave out of (75) the exp symbol. The result is a known expression for the characteristic function in terms of the fluctuation moments

$$f(\xi) = \sum K_n \frac{(-i\xi)^n}{n!}, \tag{78}$$

in which the moment M_n is replaced by the cumulant K_n. This is natural, for under condition (77) we have $M_n = K_n$ accurate to terms of order $\exp(-nu)$, i.e., accurate to the parameter (74).

Substituting in (78) the expression for K_n, whose structure is

$$K_n = C_n \nu^n e^{un^2}, \tag{79}$$

[see (9) and (72)], we obtain an asymptotic series. Here C_n are coefficients that vary slowly with n. We sum this series by an artifice similar to Borel summation, using the identity

$$\nu^n e^{un^2} = \frac{1}{(4\pi u)^{1/2}} \int_0^{\infty} w^n \exp\left[-\frac{1}{4u} \ln^2 \frac{w}{\nu} \right] \frac{dw}{w}. \tag{80}$$

We present, to be definite, the calculations for the distribution function of the density of states, for which [see (72)]

$$\nu_v = \frac{4\pi\nu_0}{g_0} \left(\frac{l}{L} \right)^2 e^{-u}, \quad C_n = C_1 \frac{1}{2n-1}, \quad C_1 = \frac{g_0 L^2}{2\pi l^2}. \tag{81}$$

Substituting (79)–(81) in (78) and changing the order of summation over n and integration with respect to w, we obtain

$$f^{add}(\xi) = \frac{C_1}{(4\pi u)^{1/2}} \int_0^{\infty} \frac{dw}{w} \int_0^{\infty} d\alpha \exp\left[\alpha - ie^{-2\alpha} w\xi - \frac{1}{4u} \ln^2 \frac{w}{\nu_v} \right]. \tag{82}$$

We have expressed here the coefficient C_n in the form of an integral with respect to the parameter α. As a result we obtain for the distribution function $f^{add}(\delta\nu)$

$$f^{add}(\delta\nu) = \frac{C_1}{(4\pi u)^{1/2}} \frac{\theta(\delta\nu)}{\delta\nu} \int_0^{\infty} d\alpha \exp\left[\alpha - \frac{1}{4u} \ln^2 \left(\frac{\delta\nu e^{2\alpha}}{\nu_v} \right) \right]$$

$$= \theta(\delta\nu) C_1 \left(\frac{u}{4\pi} \right)^{1/2} \frac{1}{\delta\nu \ln(\delta\nu/\nu_v)} \exp\left[-\frac{1}{4u} \ln^2 \left(\frac{\delta\nu}{\nu_v} \right) \right], \tag{83}$$

where $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$.

Substituting in (83) Eq. (60) for u and calculating the convolution of (76) with the Gaussian function $f^N(\delta\nu)$ we obtain for large positive $\delta\nu$ the expression given in §2 [see (7)] for the asymptotics of the distribution function $f(\delta\nu)$.

The tail of the distribution function $f(\delta\nu)$ is much weaker at negative $\delta\nu$ than at $\delta\nu > 0$. This is clear even from the fact that the odd cumulants K_{2n+1} are not small compared with the even K_{2n}. At the accuracy to which (83) was derived, $f(\delta\nu) = 0$ at $\delta\nu < 0$, as follows from the fact that the coefficient of $(-i\omega\zeta)$ in the square brackets of (82) is positive-definite. We were unable to prove this positive-definite property in general form. The coefficients C_n in (79) were calculated only in the lowest approximation in $\exp(-nu)$, i.e., in the parameter (74). It can therefore only be stated that if $f(x)$ does indeed have a tail of type (7), the ratio $f(-|x|)/f(|x|)$ at large $|x|$ is small in terms of the parameter

$$\frac{v}{x} \sim \frac{\nu_0}{\delta\nu} \left(\frac{l}{L} \right)^2 \sim \frac{1}{\delta g} \left(\frac{l}{L} \right)^2.$$

We estimate now the values of $\delta\nu$ starting with which the distribution function $f(\delta\nu)$ takes the asymptotic form (7). The total distribution function is obtained by the convolution (76) of the function $f^{add}(\delta\nu)$ (83) with the normal distribution function $f^N(\delta\nu)$, the variance being given by Eq. (52) in which, in accordance with §6, G_0 must be replaced by G. Calculating (76) by the saddle-point method, we verify that the asymptotic form (7) sets in at

$$\delta\nu > \langle (\delta\nu)^2 \rangle^{1/2} \Delta \sim \nu_0 \Delta / g, \tag{84}$$

where Δ for $d = 2$ is determined by the relation (78).

We discuss now the distribution of the fluctuations δg. In this case

$$K_n = C_n (v_g)^n e^{2un^2}, \quad v_g \sim \left(\frac{l}{L} \right)^2 e^{-3u} \tag{85}$$

and calculations similar to the ones described above yield an equation similar to (83), accurate to the prefactor of the exponential and to replacement of u by $2u$. The integral (76) leads to a logarithmically normal asymptotic (7) that is valid for $\delta g > \Delta$ (73). At small δg the distribution function becomes normal with a variance $\langle (\delta g)^2 \rangle \sim 1$ that is universal in the metallic region.

So far, only the two-dimensional case was considered in §§8 and 9. We proceed to a generalization to the case of arbitrary dimensionality $d = 2 + \varepsilon$. It suffices for this purpose to take into account the term εg in Eq. (2) for g. There is no need to consider the similar terms of Eq. (59) for X_n and Y_n, since they lead to correction of order 1 to the eigenvalues $E_n \sim n^2$ (67), (71). As a result, the only change compared with the two-dimensional case reduces to replacing u in (60) by

$$u_\varepsilon(g_0, l) = \ln \left| \frac{\varepsilon - g^{-1}}{\varepsilon - g_0^{-1}} \right| = -\ln \left| 1 - \frac{1-(l/L)^\varepsilon}{\varepsilon g_0} \right| \tag{86}$$

In accordance with the definition of u, viz., $du/d\ln(L/l) = g^{-1}$. Here, as everywhere else in this article, g_0 is the dimensionless conductance of a d-dimensional cube of size l. Equation (86) is therefore applicable in the quasi-one-dimensional or quasi-two-dimensional case. The point is that the effective dimensionality in these cases is formed over scales of the order of the transverse dimensions $a \gg l$ of the sample. Three situations should be considered: (I) quasi-two-dimensional case, (II) quasi-one-dimensional case in which the two transverse dimensions are of order a, (III) quasi-one-dimensional case in which one of the transverse dimension is smaller than or is of the order of l. If a is small compared with the localization length and $g_0 \gg 1$, the parameter u is given by

$$u = \begin{cases} g_0^{-1} \left[1 + \frac{l}{a} \ln \frac{L}{a} \right] & \text{(I)} \\ g_0^{-1} \left[1 + \frac{Ll}{a^2} \right] & \text{(II)}, \\ g_0^{-1} \left[\ln \frac{a}{l} + \frac{L}{a} \right] & \text{(III)} \end{cases} \tag{87}$$

The value of u for sufficiently large L is $u_\varepsilon(g_a, a)$, where u_ε is given by (86) and g_a is the conductance of a sample of size $L = a$. The absence of one-parameter scaling is once more manifested in the difference between (87) and (86).

Expression (7) for the asymptotics of the distribution functions remains valid at any dimensionality, if expression (86) and (87) is used for u. The value of Δ (the fluctuation scale at which the asymptotic (7) sets in) is then

$$\Delta^2 = \frac{1}{u} \ln \frac{L}{l}. \tag{88}$$

In the quasi-one-dimensional case we have $u \propto L$, meaning that the coefficient of $\ln^2 \delta g$ in (7) is proportional to L^{-1}. Just such a dependence of this coefficient on the chain length was obtained in Ref. 17 for the purely one-dimensional case.

We emphasize once more that it is impossible to describe mesoscopic fluctuation moments as the additional scaling parameters, were proposed in Refs. 51 and 52. It follows from our present results that such an approach is inadequate. On the one hand, the lower fluctuation moments ($n < g_0$) are splendidly described at $g > 1$ within the framework of one-parameter scaling. On the other, to describe high moments we need the much more elaborate scheme of §8, in which each moment is determined by a large number of scaling parameters.

We note in conclusion that near the Anderson transition the conductance fluctuations are of the order of its mean value. This fact must be taken into account in the discussion of the fundamental problems of this transition, such as whether a minimal metallic conductivity is present or not.

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APPENDIX I

Derivation of the Q-functional

We describe in this Appendix a procedure for deriving a generating functional that depends only on the slow fields $Q(\mathbf{r})$. Applying, as usual, the Hubbard-Stratanovich transformation to the functional integral (26), we reduce the action (23) to a form quadratic in the initial fermion fields:

$$S[Q,\psi]=S_0+S[h]+S[\omega]+\int\left\{\frac{i}{2\tau}\bar{\psi}Q\psi+\frac{i\pi\nu_0}{8\tau}\operatorname{Tr}Q^2\right\}d^dr, \tag{A.I.1}$$

where S_0, $S[h]$, and $S[\omega]$ are determined by expressions (22), (27) and (35), respectively, and the Hermitian field $Q(\mathbf{r})$ has the structure (24). The integral (26) is then transformed into the integral (28) over fields $Q(\mathbf{r})$ that vary slowly in space, where

$$F[Q;\mathbf{h};\omega]=-\ln\int\mathcal{D}\bar{\psi}\,\mathcal{D}\psi\exp\{iS[Q;\psi]\}. \tag{A.I.2}$$

The integral (A.I.2) is calculated by the saddle-point method. In the class of spatially homogeneous matrices $\bar{Q}$, the saddle-point condition for $\mathbf{h}=0$ and $\omega=0$ is of the form[4,5] $\bar{Q}^2=1$, $\operatorname{Tr}\bar{Q}=0$. To derive (40) from (A.I.2) we must expand the integrand in powers of the gradients of the fields $Q(\mathbf{r})$ satisfying the condition (30), and also in terms of the sources $\mathbf{h}$ and ω.

Expansion in the gradients leads in the first nonvanishing order to the standard[3-5] σ-model functional (29), (40) at $\mathbf{h}=0$ and $\omega=0$. Zero-gradient vertices containing sources $\mathbf{h}$ and ω are obtained by expanding (A.I.2):

$$F[\mathbf{h};\omega]=-\sum_{n,m}\left(\frac{i}{m_eL}\right)^{2n}\left(\frac{-1}{\pi\nu_0L^d}\right)^m$$

$$\times\frac{C^m_{2n+m}}{2n+m}\left\langle\left\langle\left[\int\bar{\psi}\mathbf{h}\nabla\psi\,d^dr\right]^{2n}\left[\int\bar{\psi}\omega\Lambda\psi\,d^dr\right]^m\right\rangle\right\rangle_{xx}. \tag{A.I.3}$$

Here $\langle\langle\ldots\rangle\rangle$ is the functional averaging (13) with weight $\exp(iS)$, and S is the action (A.I.1) in $\mathbf{h}=0$, $\omega=0$, and the field $Q(\mathbf{r})$ is assumed to be spatially homogeneous: $Q(\mathbf{r})=\bar{Q}$. The Gaussian integrals in (A.I.3) are calculated by using the Wick theorem with allowance for only connected diagrams. The paired mean values are given in the momentum representation by the expressions[48]

$$\langle\!\langle\psi(\mathbf{p})\otimes\bar{\psi}(\mathbf{p}')\rangle\!\rangle=\frac{i}{2}\mathcal{G}(\mathbf{p})\delta(\mathbf{p}+\mathbf{p}'),$$

$$\langle\!\langle\psi(\mathbf{p})\otimes\psi^T(\mathbf{p}')\rangle\!\rangle=\frac{i}{2}\mathcal{G}(\mathbf{p})\tau_2\delta(\mathbf{p}+\mathbf{p}'), \tag{A.I.4}$$

$$\langle\!\langle\bar{\psi}^T(\mathbf{p})\otimes\bar{\psi}(\mathbf{p}')\rangle\!\rangle=\frac{i}{2}\tau_2\mathcal{G}(\mathbf{p})\delta(\mathbf{p}+\mathbf{p}').$$

Here $\mathcal{G}(\mathbf{p})$ is the Green's function corresponding to the action S:

$$\mathcal{G}(\mathbf{p})=\left(\varepsilon_F-\frac{p^2}{2m}+\frac{i\bar{Q}}{2\tau}\right)^{-1}=-\left(\xi+\frac{iQ}{2\tau}\right)\left(\xi^2+\frac{1}{4\tau^2}\right)^{-1}, \tag{A.I.5}$$

where $\xi=p^2/2m-\varepsilon_F$.

Taking (A.I.4) into account, we reduce (A.I.3) at $\omega=0$ to

$$F[h]=\sum_{n\geqslant1}\frac{(-1)^n}{4n}\int\operatorname{Tr}\left\{\frac{\mathbf{h}}{L}\frac{\mathbf{p}}{m_e}\mathcal{G}(\mathbf{p})\right\}^{2n}\frac{d^d\mathbf{p}}{(2\pi)^d}L^d. \tag{A.I.6}$$

It can be seen from (A.I.6) and (A.I.5) that $F[h]$ breaks up into a sum of vertices containing $2n$ matrices $\mathbf{n}$ and an arbitrary even number, not larger than $2n$, of matrices Q. The contribution of the vertices containing three and more sources $\mathbf{h}$ in a row vanishes when (39) is differentiated. The most important, as will be shown in Appendix II, are vertices from which h^2 is also absent. For these vertices we obtain from (A.I.6)

$$F[h]=\sum_{n\geqslant1}X_n\int\operatorname{Tr}(hQ)^{2n}\frac{d^dr}{L^d}, \tag{A.I.7}$$

where $h=(h_\alpha h_\alpha)^{1/2}$,

$$X_n=g_0\left(\frac{2l}{L}\right)^{2n-d}2^{d-2}\alpha_d\frac{\Gamma(n+1/2)\Gamma(2n-1/2)\Gamma(1+1/2d)}{\Gamma\left(n+\frac{d}{2}\right)\Gamma(2n+1)} \tag{A.I.8}$$

(α_d is defined in (43)).

Equation (A.I.7) was derived for spatially homogeneous fields Q. When fields that vary in space are considered we get, besides (A.I.7), vertices containing gradients of the fields $Q(\mathbf{r})$. Calculating these vertices in lowest order in h and ∇Q we arrive at the functional (29). The contribution of the gradient vertices and the high ($n\gtrsim g_0\gg1$) fluctuation moments turns out to be small in the parameter $\exp(-n)$. The point is that none of the gradient vertices contribute, in first order, to the fluctuations of interest to us, those of homogeneous quantities.

In the study of fluctuations of the density of states we need take into account only vertices that contain powers of $\omega\Lambda Q$. The contribution of vertices containing ω^s with $s>1$ vanishes when (39) is differentiated. Using (A.I.3)–(A.I.5) we obtain for $\mathbf{h}=0$

$$F[\omega]=\sum_{m\geqslant1}Y_m\int\operatorname{Tr}(\omega\Lambda Q)^m\frac{d^dr}{L^d}, \tag{A.I.9}$$

where

$$Y_m=\frac{(-1)^m}{2\sqrt{\pi}}\frac{\Gamma\left(m-\frac{1}{2}\right)}{\Gamma(m+1)}\left(\frac{4}{\pi g_0\alpha_d d}\frac{l^d}{L^d}\right)^{m-1}. \tag{A.I.10}$$

APPENDIX II

Derivation of renormalization-group equations for additional charges

Within the framework of the standard[4,9] renormalization of the nonlinear σ model at $d=2$ we separate the "fast" and "slow" components of $Q(\mathbf{r})$:

$$Q=U^+Q_0U. \tag{A.II.1}$$

Here $Q_0(\mathbf{r})$ is the "fast" field, which is expressed with the help of the parametrization (42) in terms of the matrix T with "fast" momenta q, and $\lambda l^{-1}<q<l^{-1}$ (λ is a scale factor). The "slow" unitary matrices U reconstruct the "slow" field $\bar{Q}(\mathbf{r})$ in the form

$$\bar{Q}(\mathbf{r})=U^+\Lambda U. \tag{A.II.2}$$

The renormalized functional is obtained after integration over $Q_0(\mathbf{r})$:

$$F[\bar{Q}]=-\ln\int\mathcal{D}Q_0\exp\{-F[Q]\}.$$

To carry out the renormalization-group transformations (A.II.3) in the one-loop (linear in $t\propto g^{-1}$) approximation, the integrand in (A.II.3) must be written in the form $\exp(-F+F_0)\exp(-F_0)$ and $F-F_0$ must be expanded up to terms quadratic in W_0 using the parametrization (42) for Q_0. The functional F_0, just as in perturbation theory, is given by (44), in which, however, the matrices $W(q)$ are replaced by matrices $W_0(q)$ that contain only "fast" momenta.

Consider the changes produced in the vertex (31) by the renormalization-group transformations. After separating the "fast" fields in accordance with (A.II.1), we get

$$\Phi_n=X_n\int\varphi_n(r)\,d^dr,\quad \varphi_n\equiv\operatorname{Tr}(\hbar Q)^{2n},\quad \hbar\equiv UhU^+. \tag{A.II.4}$$

We expand the density φ_n of the functional Φ_n up to terms quadratic in W_0

$$\varphi_n=\tilde{\varphi}_n+2n\operatorname{Tr}[(\hbar\Lambda)^{2n}W_0]+\sum_{j=0}^{2n-1}\operatorname{Tr}[(\hbar\Lambda)^jW_0(\hbar\Lambda)^{2n-j}W_0]. \tag{A.II.5}$$

The term $\tilde{\varphi}_n$ in (A.II.5) is independent of W_0 and reformats the vertex Φ_n in terms of slow variables $\tilde{\varphi}_n=\operatorname{Tr}(h\bar{Q})^{2n}$. The term linear in W_0 makes no contribution to the mean value. Averaging in (A.II.5) the term quadratic in W_0 by using (47), we get

$$\langle\Phi_n\rangle_0=\tilde{\Phi}_n-nX_ng^{-1}\ln\lambda^{-1}\int\left\{\tilde{\varphi}_n+\sum_{j=1}^{n-1}\tilde{\varphi}_j\tilde{\varphi}_{n-j}\right\}d^2r. \tag{A.II.6}$$

We have left out of (A.II.6) the terms proportional to N, which make no contribution in the replica limit $N=0$, as well as the terms with a number of Q matrices smaller than $2n$. Their contribution to the renormalization-group equations will be analyzed below.

Renormalizing the vertices containing products of φ, we arrive ultimately at a functional containing all the vertices of type (57). In the one-loop approximation it is necessary to expand the factors φ_m in each of the vertices (57) in accordance with (A.II.5) and, retaining in the product $(\varphi_l)^{s_l}\ldots(\varphi_m)^{s_m}\ldots$ only the terms quadratic in W_0, carry out the averaging using (47) and (48). As a result we get the contribution of any of the vertices (57) to the renormalization of the functional

$$\left\langle X_n^s\int\{\varphi_l^{s_l}\ldots\varphi_m^{s_m}\ldots\}d^2r\right\rangle_0$$

$$=X_n^sg^{-1}\ln\lambda^{-1}L\int\{\varphi_l^{s_l}\ldots\varphi_m^{s_m}\ldots\}d^2r,$$

$$\sum_{n\geqslant1}m\varphi_m\frac{\partial}{\partial\varphi_m}+\sum_{m,l\geqslant1}\Big[(m+l)\varphi_m\varphi_l\frac{\partial}{\partial\varphi_{m+l}}$$

$$+4ml\varphi_{m+l}\frac{\partial}{\partial\varphi_m}\frac{\partial}{\partial\varphi_l}\Big]. \tag{A.II.7}$$

Differentiation of (A.II.7) with respect to $\ln\lambda^{-1}$ leads to the system of linear differential equations (61), in which we have changed to the more convenient form (62) of the operator L_{ij}.

As noted in Appendix I, the derivation of a generating functional gives rise, besides (41), to vertices in which the number of Q matrices is less than that of h matrices. An example is the vertex

$$X_n'\int\operatorname{Tr}\{h^2Q(hQ)^jh^2Q(hQ)^{2n-j-4}\}d^dr.$$

Vertices containing less than $2n$ fields Q are generated in the renormalization (57) (they were left out of (A.II. is important that these vertices make no contribution renormalization of the charges at the vertices (57): not be changed into $hQhQ$ by a renormalization-group formation.

The renormalization of the charges X_n' at the v (A.II.8) is described by renormalizaiton-group equat which the charges X_n^s at the vertices (57) enter as in geneous terms. It is important that the eigenvalues o equations, which are determined by the number of fi turn out to be smaller than (67). Therefore their increm large n is determined by the inhomogeneous terms X_n^s also given by (68). Consequently, allowance for these would change only the prefactor of the exponential equation $K_n\propto\exp(un^2)$ for the growth of the high ation moments.

[1)]Here and elsewhere g means the mean value of the dimensionle ductance, and $\delta g\equiv g(G-\langle G\rangle)/\langle G\rangle$.

[2)]We are considering the conductance of a cube of volume L^d external-field frequency and at $T=0$.

[3)]$\hbar=1$ here and everywhere else, except in the final results.

[4)]The terms containing R^+ in the right-hand sides of these identitie spond to Cooper contributions, and the remaining ones to diffusi tributions. Equations (47) and (48) are valid also in the presen weak magnetic field if the Cooper contributions are discarded elements of matrices P and R are assumed to be complex numbe

[5)]Note that certain topologically possible diagrams, such as, e.g., d 3i, do not appear at all in the chosen parametrization (42). The su diagrams of a given order does not depend, of course, on the param tion, but the coefficients preceding the individual diagrams do d Thus, in the usual crossover technique, to which the parametr $Q^{11}=(I-Q^{12}Q^{21})^{1/2}$ corresponds,[7] the diagram 3i and those sim it differ from zero, while the diagrams 3d and 3h vanish.

[6)]We are grateful to V. I. Yudson who pointed out this Ansatz tha back to the theory of representations of symmetric groups (see, e.g 49). Note that the relation such as (67) was first obtained by We as an anomalous dimensionality of operators connected with th ments of a local density of states.

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Mössbauer isomer shifts of ^{119}Sn in metals and alloys

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The Mössbauer isomer shifts (S) of ^{119}Sn were measured in homogeneous metal matrices, in certain binary alloys, and in intermetallides. The main property that determines the behavior of S for Sn atoms in metallic systems is the locality property, which reflects the dominant contribution of the interaction with the nearest neighbors and the observed changes of the electron density. A linear correlation between S and the electronegativity φ_M of the matrix was obtained for Sn atoms admixtures in pure metals. The drastic weakening of the dependence of S on φ_M for transition metals is due to the decrease of the s-component of the wave function as a result of sd hybridization. The nonlinearity of the dependence of S on the composition in binary alloys is interpreted with the aid of a model in which account is taken of the inhomogeneous electron-density distribution, due to the difference between the energies of Sn-atom bonds with neighboring atoms of various types. We discuss the significance of these results to the interpretation of data on magnetic hyperfine interaction of Sn atoms with metallic magnetics.

1. INTRODUCTION

One of the most important parameters measured in Mössbauer spectroscopy is the isomer shift (IS) of a resonance line. The IS is proportional to the electron density in the region of the nucleus. The value of the IS is highly sensitive to changes of the configuration of the atom's valence electrons. The IS is therefore widely used to study the electron structure of solids. By measuring the IS of a metallic system it is possible to investigate the electron-density distribution changes that reflect the features of the interaction between the atoms when an alloy or an intermetallic compound is formed (see, e.g., Refs. 1 and 2).

The study of the IS of ^{119}Sn impurity atoms in homogeneous metallic matrices is the subject of many studies made mainly during the early development of Mössbauer spectroscopy. (The main results of these studies and a bibliography can be found, for example, in Flinn's review[2].) Certain relations were established, in the form of correlations between the magnitude of the IS and such matrix-metal properties as the atomic volume, the compressibility, the Debye force, and others. Unfortunately, none of these correlations have been thoroughly interpreted. The question of the factors that determine the value of the IS for Sn atoms in metals remained unanswered even empirically. It is easily seen that by a mere determination of the above-mentioned correlations contributes little to the solution of this problem. The main characteristic properties of pure metals are known to depend in like fashion on the place of the element in the periodic table, so that all these properties strongly correlate with one another. The existence of a correlation between the IS and some property of a matrix metal is therefore almost trivial and reflects in the main only a general interrelation between various properties of metals. It should also be noted that sufficiently pronounced correlations were obtained only for Sn impurity atoms in nontransition metals. This is partially attributed to the incomplete accuracy of the experimental data, and also to the fact that most properties of transition-element matrix metals taken as a group vary in a relatively narrow range.

The isotope ^{119}Sn is widely used in Mössbauer spectroscopy as a probe to study the electronic and magnetic structures of metallic systems. The measured values of the IS contain important information both on the properties of the system and on the features of the interaction between the Sn impurity atom and the matrix atoms. Unfortunately, an appreciable (and the most important) part of this information cannot be extracted from experimental data, since there are no specific data on the relation between the observed values of the IS and the parameters of interest. An experimental determination of the main factors that determine the value of the IS for Sn atoms in metallic systems is therefore most urgent. Interest in this problem is increasing in view of the recently observed correlation between IS and hyperfine fields for Sn impurity atoms in gadolinium intermetallides.[3] It was shown that in a number of cases the observed parameters of the magnetic hyperfine interaction reflect directly the features of the chemical interaction. One can expect a correct interpretation of the IS data to be useful for elucidation of the nature of the mechanism that transfers the spin density from magnetic atoms to nonmagnetic ones in metallic magnetics.

We have carried out systematic measurements of the IS for ^{119}Sn in metals, binary alloys, and some intermetallic compounds, having crystal structures typical of metallic systems. All the IS measurements were made under identical conditions: at 77 K relative to a $CaSnO_3$ source kept at room temperature. The measurements were made with spectrometers operating in a constant-acceleration regime. In most cases, a double spectrometer[4] was used, which permitted highly accurate measurements of the relative shifts of the Mössbauer lines. As a rule, the absolute values of the IS were measured with accuracy not worse than 0.02 mm/s; the relative measurements of the IS in binary alloys were measured

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