$$F_\pm(|x|)=-\frac{ih^2\eta\Lambda k}{q_\pm+r}\exp(i|x|q_\pm+ik|x|\eta^2)$$
$$\times\left[\Gamma\left(-\frac{1}{2},\ ik|x|\eta^2\right)+4\pi^{1/2}\theta(\varkappa-n)\right],$$
$$\mathrm{Re}[(i\eta^2)^{1/2}]>0,\quad q_\pm=k(1\mp\sin\psi).$$

The step function satisfies $\theta(y)=1$ when $y>0$ and $\theta(y)=0$ when $y<0$, Γ is an incomplete gamma function, $\eta=i(2\varepsilon)^{-1/2}$,

$$\Phi(|x|)=h^2\beta[\mathrm{ci}(\beta|x|)\cos(\beta|x|)+\mathrm{si}(\beta|x|)\sin(\beta|x|),$$
$$\beta=4k\varkappa\cos^2\psi(\Lambda/r),\quad \beta/k\ll1,$$

and ci and si are the cosine and sine integrals.

Two limiting cases are of special interest. The first, when $a\gg b$, where $a=\mathrm{Re}\,\eta$ and $b=\mathrm{Im}\,\eta$, corresponds to the existence of surface electromagnetic waves (SEW). When the inequalities $a^2k\,|x|\gg1$ and $b^2k\,|x|\ll1$ hold, we find

$$\mathrm{Re}\,F_\pm(|x|)\approx\frac{h^2\Lambda ka}{q_\pm+r}\left[4\pi^{1/2}\exp(-2k|x|ab)\sin(q_\pm|x|+a^2k|x|)-\frac{\sin(q_\pm|x|\pm\pi/4)}{(a^2k|x|)^{1/2}}\right].\qquad(10)$$

For the inverse situation, when $b\gg a$ and the inequalities $b^2k\,|x|\gg1$ and $a^2k\,|x|\ll1$ hold, we obtain

$$\mathrm{Re}\,F_\pm(|x|)\approx\frac{h^2\Lambda kb}{q_\pm+r}\frac{\sin(q_\pm|x|+\pi/4)}{(b^2k|x|)^{1/2}}.\qquad(11)$$

The theoretical results thus obtained make it possible to explain the experimental results given above. First of all we note that, in agreement with experiment (see Fig. 1) and expressions (9) and (10), the structures which form are related to the excitation of SEW, since the period $d_\pm$ of a profile at small angles of incidence differs from the wavelength of the incident radiation. As the increase of the incidence angle the picture becomes asymmetric with respect to the initial perturbation (see Figs. 3 and 4), which is explained theoretically [see formula (9)] by the presence of the function $\theta(\pm x)$, multiplied by the function $F\pm(|x|)$, which oscillates with different periods on opposite sides of initial perturbation. Note that the function $\Phi(|x|)$ that enters Eq. (9) does not change significantly the type of structure, since it is a sufficiently slowly oscillating, weakly decaying function.

From Eq. (10) follows an interesting fact, namely, that the periodic structure is localized over distances $|x|\sim(2kab)^{-1}$, since at large values of the power density Q the quantity $\Lambda/(q_\pm+r)$ depends weakly on Q [$\Lambda\sim Q$, $r\sim Q$ (see Refs. 3, 6, and 8)]. This explains the experimental dependence shown in Fig. 2. The estimates obtained by comparing theory with experimental data give for the quantities a and b the values $a=0.3$ and $b=0.03$, what agrees well with the inequality $a\gg b$.

Let us note that, as follows from this theoretical analysis, excitation of SEW is impossible when $b>a$, but in this case the structures can develop with a period which is determined only by the wavelength and angle incidence of the laser radiation. These structures can be formed as a result of interference of volume electromagnetic waves.

We emphasize that all the experimental results given in this work can be explained within the framework of linear theory. We think that the fact that the periodic structures do not spread to a larger area with further repeated exposures to laser radiation also supports the above statement that linear theory describes the major effects in the formation of surface structures.

In conclusion, we note that the experimental and theoretical results obtained in this work demonstrate the important role of initial perturbations in the forming periodic surface structures.

The authors express deep gratitude to A. A. Denisov and N. A. Timonina for their help with the experiments.

[1] S. I. Anisimov, A. M. Gol'dberg, B. A. Malomed, and M. I. Tribel'skii, Dokl. Akad. Nauk SSSR **262**, 1117 (1982) [Sov. Phys. Dokl. **27**, 130 (1982)].
[2] V. I. Emel'yanov, E. M. Zemskov, and V. N. Seminugov, Kvantovaya Elektron. (Moscow) **10**, 2389 (1983) [Sov. J. Quantum Electron. **13**, 1556 (1983)].
[3] V. N. Anisimov, V. Yu. Baranov, L. A. Bol'shov *et al.*, Poverkhnost' 7, 138 (1983).
[4] V. I. Emel'yanov and V. N. Seminogov, Zh. Eksp. Teor. Fiz. **86**, 1026 (1984) [Sov. Phys. JETP. **59**, 598 (1984)].
[5] D. C. Emmony, R. P. Howson, and L. J. Willis, Appl. Phys. Lett. **23**, 598 (1973).
[6] M. K. Kochergina, M. N. Libenson, V. S. Makin, and S. D. Putkov, p. 381, Abstracts of Papers at the Fifth All-Union Conference on Interaction of Optical Radiation with Materials, Nauka, Leningrad, 1981.
[7] A. M. Prokhorov, A. V. Svakhin, V. A. Sychugov *et al.*, Kvantovaya Elektron. (Moscow) **10**, 906 (1983) [Sov. J. Quantum Electron. **13**, 568 (1983)].
[8] B. I. Makshantsev, N. F. Pilipetski, Appl. Phys. **A36**, 205 (1985).
[9] E. Keilmann and Y. H. Bai, Appl. Phys. **A29**, 9 (1982).
[10] S. I. Anisimov, Ya. A. Imas, G. S. Romanov, and Yu. V. Khodyko, Deistviye izlucheniya Bol'shoi moshchnosti na metaly (Effects of high-power radiation on metals), Nauka, Moscow, 1970.
[11] E. L. Feinberg, In: "Issledovaniya po rasprostraneniyu radiovoln" (Investigations of radiowave propagation), Avademizdat, Moscow-Leningrad, p. 97, 1948.
[12] L. D. Landau and E. M. Lifshitz, p. 43, 130, Teoriya uprugosti (Theory of Elasticity), Fizmatgiz, M., 1965 [English trans. Pergamon Press, London, 1959].
[13] D. Key and T. Loby, Handbook of Experimental Physicist (Russ. trans. IIL, Moscow, 1949).
[14] Brief Technical Physics Handbook, V. 1, Fizmatgiz, Moscow, 1960.

Translated by A. Petelin

Repulsion of energy levels and conductivity of small metal samples

B. L. Al'tshuler and B. I. Shklovskiĭ

B. P. Konstantinov Institute of Nuclear Physics, Academy of Sciences of the USSR, Leningrad

(Submitted 22 January 1986)
Zh. Eksp. Teor. Fiz. **91**, 220–234 (July 1986)

A metal sample of size L is considered in the one-electron approximation for the case in which the conduction electrons move in a random impurity potential. The fluctuations in the number of levels in an energy band of width E due to random variations in the potential are studied, and it is shown that the fluctuations are described by Dyson's expression when $E\ll E_c$, where $E_c=\hbar D/L^2$ and D is the diffusion coefficient. For $E\gg E_c$ the fluctuations in the number of levels are much larger than those found by Dyson. Thus for high energies the Wigner-Dyson postulates are not satisfied by an ensemble of metal samples. The relationship between the fluctuations in the sample conductivity and the fluctuations in the level density is analyzed, and the conductivity fluctuations are found to consist of two independent contributions. The first is determined by the fluctuations of the level number in a band of width max $\{E_c,T\}$ centered in the Fermi level (T is the temperature). The second contribution is due to fluctuations of the diffusion coefficient. Both contributions are of the same order of magnitude when $T\ll E_c$. For $T\gg E_c$ the result depends on the dimensionality d of the sample. For $d=3$ the contributions from the state density and from the diffusion coefficient remain comparable, while the fluctuations in the diffusion coefficient are dominant for $d=1$ and 2.

1. INTRODUCTION

Fluctuations of the residual conductivity in small conductors have recently attracted both experimental[1–3] and theoretical interest.[4–9] The fluctuations can give rise to reproducible aperiodic oscillations in the conductivity when the magnetic field varies. The theory, which seems to agree with experiment, predicts that at zero temperature the reciprocal G of the sample resistance (i.e., the conductance) differs from its average value by an amount $\delta G\sim e^2/\hbar$ (it also depends on the size and shape of the sample):

$$\langle\delta G^2\rangle\equiv\langle G^2\rangle-\langle G\rangle^2\approx(e^2/\hbar)^2,\qquad(1)$$

where $\langle\ldots\rangle$ denotes an average over all realizations of the random potential.

According to (1), the relative magnitude $\delta G/G$ of the fluctuation is independent of the sample size L in the two-dimensional case ($d=2$) and is proportional to $1/L$ for $d=3$. We note that conductivity fluctuations caused, e.g., by fluctuations in the impurity concentration, should fall off as $L^{-d/2}$. From this viewpoint the fluctuations described by Eq. (1) would appear to be anomalously large.

However, these fluctuations can also be approached from another point of view. Thouless[10] showed that the conductivity of a sample of size L is determined by precisely defined one-electron levels which lie in an energy band of width $E_c=D\hbar/L^2$ centered in the Fermi level (D is the electron diffusion coefficient and L^2/D is the characteristic time for an electron to diffuse through the sample). It is easy to verify that

$$\langle G\rangle\approx\frac{e^2}{\hbar}\langle N(E_c)\rangle,\qquad(2)$$

where $N(E)$ is the number of levels in a band of width E. It is reasonable to assume that the fluctuations in the conductance are due to the fluctuations $\delta N(E_c)\equiv N(E_c)-\langle N(E_c)\rangle$ in the number of levels $N(E_c)$:

$$\delta G/\langle G\rangle\approx\delta N(E_c)/\langle N(E_c)\rangle.\qquad(3)$$

Using (1) and (2), we readily find from (3) that

$$\delta N(E_c)\approx1,\qquad(4)$$

whereas one would expect the result $\delta N\approx N^{1/2}$ for a disordered system. Thus from this point of view, the conductivity fluctuations (1) are anomalously weak.

The fact that the level number fluctuations for a given energy band are small compared to $N^{1/2}$ indicates that there must be a repulsion between the levels. Since the appearance of Wigner's work,[11] this phenomenon has been actively studied for systems describable by a matrix Hamiltonian, all of whose elements fluctuate about zero in the same way with a large amplitude. This problem is treated, e.g., in the statistical theory of nuclear spectra.[12,13] Arguments based on level repulsion were first applied to small metal particles in Ref. 14.

Most of this work (see the recent paper Ref. 15 and the bibliography cited there) is concerned with how the repulsion alters the distribution function for the distance between two adjacent levels. However, as suggested by the above discussion, we are primarily interested in the different problem of calculating the fluctuations in the number of levels $N(E)$ in an energy band of width E when the average number of levels $\langle N(E)\rangle$ in the band is large. This problem was first discussed by Dyson[16] and was subsequently analyzed more fully in Ref. 17. Dyson showed that

$$\langle[\delta N(E)]^2\rangle=\frac{2ks^2}{\pi^2\beta}\ln\langle N(E)\rangle,\qquad(5)$$

0038-5646/86/070127-09$04.00

where the number β is different for each of the three ensembles of random Hamiltonians introduced by in Ref. 16 (β is equal to 1,2, and 4 for the orthogonal, unitary, and symplectic ensembles, respectively, which we recall are characterized by different symmetry constraints). According to Ref. 14, the orthogonal ensemble describes metal particles when there is no magnetic field or spin-orbit scattering; the unitary ensemble is appropriate in a magnetic field, while the sympletic ensemble describes the case of strong spin-orbit scattering. In Eq. (5) $N(E)$ is the total number of states (countted with allowance for a possible s-fold degeneracy which is not removed by the random Hamiltonian). The simplest example is two-fold spin degeneracy in an orthogonal ensemble.

The quantity k in (5) is equal to the number of the noninteracting series of levels; indeed, levels with different precisely defined quantum numbers do not interact with one another. It is clear that in this case the mean-square fluctuations for the level numbers in the different series can simply be added.

It is clear from (5) that

$$|\delta N(E)| \ll (\langle N(E)\rangle)^{1/2},$$

i.e., the level number fluctuations are strongly suppressed by the repulsion, although they are still larger than predicted by Eq. (4). In other words, substitution of (5) into (3) leads to conductance fluctuations greater than predicted by (1).

We will resolve this apparent contradiction in the next section, where the meaning of Dyson's result (5) will be discussed in detail. We will find that Eq. (5) applies only to an isolated sample, i.e., to specimens that retain all their electrons and therefore have unbroadened energy levels. On the other hand, Eq. (1) was derived for "open" samples in contact with two perfect conductors. The electron energy levels in this case are broadened by an amount $\sim E_c = \hbar D/L^2$.

In Sec. 3 the impurity diagram technique is used to calculate the fluctuations of the number of levels in a band of width E in a metal sample. For an isolated sample and small $E (E < E_c)$, the result coincides with Eq. (5). We will see how the fluctuations δN get smaller when the sample is "opened" and Eq. (5) gradually merges into (4).

The situation is very different when $E \gg E_c$. In this case the fluctuations $\delta N(E)$ for isolated and open samples are comparable in order of magnitude and much larger than (4) and (5),

$$\langle[\delta N(E)]^2\rangle = \frac{c_d k s^2}{\beta}\left(\frac{L}{L_E}\right)^d = \frac{c_d k s^2}{\beta}\left(\frac{E}{E_c}\right)^{d/2}, \qquad (6)$$

where d is the dimensionality of the sample, $L_E = (D\hbar/E)^{1/2}$,

$$c_1 = 2^{1/2}/\pi^2, \quad c_2 = 1/4\pi^2, \quad c_3 = 2^{1/2}/6\pi^3. \qquad (6a)$$

Thus for wide bands $E \gg E_c$, metal samples do not obey the conditions postulated by Wigner and Dyson.

Equation (6) implies that as E increases, the system of levels becomes less "rigid" and $\langle[\delta N(E)]^2\rangle$ depends on the sample volume L^d in the same way as if no repulsion were present between the levels. However, the repulsion still cannot be neglected, because the mean-square magnitude of the fluctuations and its dependence on E do not obey Poisson statistics. Indeed, (6) implies that

$$\langle[\delta N(E)]^2\rangle / \langle N(E)\rangle = e^2/\hbar G_E \ll 1,$$

where G_E is the conductance of a cube of side L_E. The physical significance of Eq. (6) will become clearer from the qualitative derivation given in Sec. 2.

In Sec. 4 we quantitatively analyze the relationship between the conductance and state-density fluctuations. We will see that for $T \ll E_c$ the corresponding relative magnitudes are indeed comparable. Nevertheless, one-third of the magnitude of the mean-square conductance fluctuations is due to fluctuations in the density of states; the remaining two-thirds is due to fluctuations in the electron diffusion coefficient. For temperatures $T \gg E_c$, the result depends on the dimension d. For $d = 3$, the state density fluctuations account for 1/6 of the total $\langle\delta G^2\rangle$, while for low dimensions the quantity $\langle\delta G^2\rangle/\langle G\rangle^2$ turns out to be systematically greater than $\langle[\delta N(T)]^2\rangle/\langle N(T)\rangle^2$:

$$\frac{\langle\delta G^2\rangle}{\langle G\rangle^2}\,\frac{\langle N(T)\rangle^2}{\langle[\delta N(T)]^2\rangle} \approx \begin{cases} \ln T/E_c, & d=2 \\ (T/E_c)^{1/2}, & d=1 \end{cases}. \qquad (7)$$

Thus for $T \gg E_c$ and $d \leqslant 2$, the fluctuations in the conductance are due almost entirely to fluctuations in the diffusion coefficient, and Eq. (3) does not even give the correct order of magnitude.

It will also be shown in Sec. 4 that the state-density and conductance fluctuations are independent, i.e.,

$$\langle\delta N(T)\,\delta D\rangle = 0,$$

where D is the diffusion coefficient.

2. QUALITATIVE ANALYSIS

The level repulsion is easiest to understand for the case of a 2×2 matrix Hamiltonian[11] with independent random-variable elements H_{ij}. In this case the probability density for the system to have the levels ε_1 and ε_2 is equal to

$$P_2(\varepsilon_1, \varepsilon_2) = \int \delta\{\varepsilon_1 - \varepsilon_2 - [(H_{11}-H_{22})^2 + H_{12}^2]^{1/2}\}\,\delta(\varepsilon_1 + \varepsilon_2 - H_{11} - H_{22}) \times f(H_{11}) f(H_{22}) f(H_{12})\, dH_{11}\, dH_{22}\, dH_{12}, \qquad (8)$$

where $f(H_{ij})$, the distribution function of H_{ij}, varies on a scale H_0. For real H_{12} (this corresponds to an orthogonal ensemble) and $|\varepsilon_1 - \varepsilon_2| \ll H_0$, we find from (8) that

$$P_2(\varepsilon_1, \varepsilon_2) \propto |\varepsilon_1 - \varepsilon_2|/H_0^3. \qquad (9)$$

For a unitary ensemble, H_{12} is complex and Re H_{12} and Im H_{12} are independent random variables, and

$$P_2(\varepsilon_1, \varepsilon_2) \propto |\varepsilon_1 - \varepsilon_2|^\beta / H_0^{2+\beta} \qquad (9a)$$

with $\beta = 2$. Relation (9a) with $\beta = 4$ can be derived similarly for a symplectic ensemble.

The multilevel distribution function was first derived by Wigner[11] for an orthogonal ensemble with

$$f(H_{ij}) = \frac{1}{\pi^{1/2} H_0}\exp\left\{-\frac{H_{ij}^2}{2H_0^2}\right\}. \qquad (10)$$

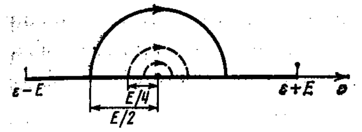

FIG. 1.

The result is

$$P_N\{\varepsilon_i\} = B_N H_0^{-N(N+1)/2} \prod_{i>j} |\varepsilon_i - \varepsilon_j| \exp\left(-\sum_i \varepsilon_i^2/2H_0^2\right). \qquad (11)$$

The product of the $|\varepsilon_i - \varepsilon_j|$ for all $i > j$ describes the repulsion between the pairs of levels in the system; its origin can be explained as follows. Dimensional analysis shows that the function $P_N\{\varepsilon_i\}$ must contain a polynomial of degree $N(N-1)/2$. On the other hand, this polynomial must vanish as $|\varepsilon_i \varepsilon_j|$ when $\varepsilon_i \to \varepsilon_j$, because only a pair of levels describable by a 2×2 Hamiltonian needs to be considered in this limit. It follows that the polynomial must be of the form $\Pi_{i>j}|\varepsilon_i - \varepsilon_j|$. A more rigorous derivation of (11) can be found in Ref. 12, pp. 260–262. The generalization of (11) valid for all three types of ensembles is

$$P_N\{\varepsilon_i\} = C_N \prod_{i>j} |\varepsilon_i - \varepsilon_j|^\beta \exp\left(-\sum_i \varepsilon_i^2/2H_0^2\right). \qquad (12)$$

We will find it useful to exploit Dyson's analogy with electrostatics as formulated in Ref. 16. If we write (12) in the form

$$P_N\{\varepsilon_i\} = C_N \exp\left\{-\beta\sum_{i>j} \ln(1/|\varepsilon_i - \varepsilon_j|) - \sum_i \varepsilon_i^2/2H_0^2\right\}, \qquad (13)$$

it is clear that $P_N\{\varepsilon_i\}$ is the probability density for N particles at temperature β^{-1} to lie at the points ε_i on the energy axis if the particles repel one another by the "two-dimensional Coulomb" law

$$U(\varepsilon_i - \varepsilon_j) = \ln(1/|\varepsilon_i - \varepsilon_j|). \qquad (14)$$

and are confined by a quadratic potential $\varepsilon_i^2/2H_0^2\beta$ near the point $\varepsilon = 0$. Dyson[16] proposed an alternative method for confining a Coulomb gas, in which the ε axis is regarded as closing back on itself to form a circle. Since we will be interested in the density fluctuations of a Coulomb gas in an energy interval small compared to the confinement region (i.e., to the total width of the spectrum), we need not specify the confinement mechanism. The only important thing is that the "gas" be in a state of mechanical equilibrium.

To estimate the fluctuation $\delta N(E)$ in the number of levels in an energy interval E, we take an arbitrary point ε and consider a small neighborhood of length $2E$, which we split into halves (Fig. 1). Let us calculate the energy W needed to take the left-hand half into the right-hand half, i.e., to take the δN levels from the segment $[\varepsilon - E, \varepsilon]$ into the segment $[\varepsilon, \varepsilon + E]$. This is the energy required to form additional charges of $\mp\delta N$ in the two segments. We need consider only the interaction of these charges with one another, because in the unperturbed state the total force acting on each particle is equal to zero. (In the context of a Coulomb gas, the confining field acts as a compensating background in Wigner's approach and one can speak of particles and holes.)

If we calculate the difference between the interaction energies for the charges $\pm\delta N$ in the final and initial states, we find that the required energy is

$$W = -(\delta N)^2 \ln(1/E) - [-(\delta N)^2 \ln(2/E)] \approx (\delta N)^2. \qquad (15)$$

For a typical fluctuation δN, W should be comparable to the temperature β^{-1}; it follows that

$$\langle[\delta N(E)]^2\rangle_E \approx 1/\beta \approx 1. \qquad (16)$$

Formula (14) gives the size of fluctuations which are homogeneous over the energy scale E, and this is emphasized in (16) by the presence of the subscript E. However, fluctuations in the number of levels within the segment $[\varepsilon - E, \varepsilon]$ result not only from fluctuations of scale E but also from fluctuations of scale $E/2, E/4, \ldots, E/2^\eta$, which may be regarded as independent (see Fig. 1). The minimum energy scale $E/2^\eta$ below which the above approach becomes invalid is comparable to the mean distance $\Delta \equiv E/\langle N(E)\rangle$ between the levels. Therefore,

$$n = \ln(E/\Delta)/\ln 2 = \ln\langle N(E)\rangle/\ln 2. \qquad (17)$$

Formula (16) has the remarkable property that there is no dependence on the subscript E. This implies that all the energy scales contribute equally to $\langle[\delta N(E)]^2\rangle$,

$$\langle[\delta N(E)]^2\rangle = \int \langle[\delta N(E)]^2\rangle_{E'}\, d\ln(E/E') = n\langle[\delta N(E)]^2\rangle_E. \qquad (18)$$

Upon substituting (16) and (17) into (18), we recover Dyson's result (5) up to a numerical factor.

So far we have considered the number of levels in an energy band with well-defined boundaries. If the band center lies at energy ε, the number of levels is equal to

$$N_\varepsilon(E) = \int_{-\infty}^{\infty} d\varepsilon'\, \theta\left[\frac{E^2}{4} - (\varepsilon - \varepsilon')^2\right] \sum_i \delta(\varepsilon' - \varepsilon_i), \qquad (19)$$

where $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x > 0$. Bands with blurred edges are clearly of interest in physical applications. Formally, this amounts to replacing the θ-functions in (19) (solid curve in Fig. 2) by a smooth function $F(\varepsilon - \varepsilon')$ with edges of width $\sim\gamma$ (dashed curve). In this case the fluctuations $\delta N(E)$ clearly depend only on the amount of charge transported over energy distances greater than γ. Rather than using Eq. (17) for n in (18), we must set $n \approx \ln(E/\gamma)$, whence

$$\langle[\delta N(E)]^2\rangle = \frac{2ks^2}{\pi^2\beta}\ln\frac{E}{\gamma}. \qquad (20)$$

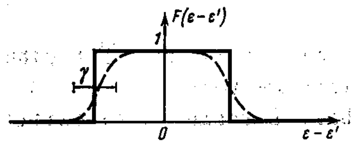

FIG. 2.

We note that the above qualitative analysis does not suffice to determine the numerical coefficient in Eq. (20); the value given here follows from the exact calculation in Sec. 3 for a metal sample.

Equation (20) can also be interpreted somewhat differently. We first consider a band with sharp edges but allow for the finite width γ of the levels ε_i (the broadening could be due, e.g., to inelastic collisions). The broadening of the levels can be allowed for in (19) by making the replacement

$$\delta(\varepsilon-\varepsilon_i)\to\frac{1}{2\pi}\frac{\gamma}{(\varepsilon-\varepsilon_i)^2+\gamma^2/4}. \tag{21}$$

One must now work with the total number of states rather than consider the number of levels. Of course, Eq. (20) also describes the fluctuations in the number of states. We note that Eq. (20) reduces to (5) if $\gamma\approx\Delta$ and to (4) if $\gamma\approx E$.

The above results can be used to analyze the conductivity of a small metal sample. As stated in the Introduction, the conductance is proportional to the number of levels in a band of width E_c. However, it is clear that the band cannot have sharply defined boundaries. The Thouless approach,[10] which is based on analyzing how a level shifts when the boundary conditions change, implies that the boundaries of the energy band are blurred by an amount comparable to the band width E_c, itself. Equation (20) then implies (4), and the repulsion of the levels causes fluctuations of magnitude $\sim e^2/\hbar$ in the conductance. The same result is obtained if an open sample is considered; in this case the level width γ is again comparable to E_c because electrons can escape from the metal, and (20) again reduces to (4).

It makes sense in principle to consider a "weakly open" sample, for which the contacts to the other conductors have a small transparency η. In this case formula (20) applies with $\gamma=\eta E_c\ll E_c$.

We have thus far examined some consequences of level repulsion in systems obeying the Wigner-Dyson postulates. For such systems the result (20) is valid whenever the band width E is much less than the total width of the spectrum. As was mentioned in the Introduction, an exact calculation (Sec. 3) shows that the Wigner-Dyson approach applies to metal samples only for sufficiently narrow bands $E\lesssim E_c$. In the opposite limit $E\gg E_c$ the level number fluctuations obey Eq. (6), which differs radically from (20). To account for this difference, we note that from our perspective the chief assumption underlying the Wigner-Dyson result is that all the matrix elements have a scatter of the same order of magnitude. This is valid for a metal sample if a narrow energy band $E\lesssim E_c$ is considered. In this case an electron can diffuse through the entire sample during a time of order $\hbar/E$, so that all the matrix elements H_{ij} are comparable to E_c in order of magnitude. By contrast, for a wide band an electron can diffuse through only a portion of the sample during the time $\hbar/E$:

$$L_E=(\hbar D/E)^{1/2}\ll L. \tag{22}$$

This suggests breaking the whole sample up into small cubes with sides of length L_E. The states contained in a band of width E can be used to form wave packets that are localized within the cubes. Clearly, the matrix elements relating wave packets from a single cube differ dramatically in order of magnitude from the matrix elements relating packets from two cubes separated by a distance $L\gg L_E$. It is therefore reasonable to adopt an approximation in which each cube has its own system of levels, with no repulsion between levels in different cubes. By (20), the fluctuation in the number of levels in each cube is of order unity. (According to the definition (22), the width is $\sim E$ in an open cube of side L_E.) The quantity $\langle(\delta N(E))^2\rangle$ for the whole sample is therefore roughly equal to $(L/L_E)^d$, the number of cubes (see Eq. (6)).

We see that $(L/L_E)^d$ in this problem is analogous to the number k of noninteracting series of levels that appears, e.g., in (5). For metal samples with a wide energy band, the effective number of series depends on the band width E.

Equation (6) is of course valid only for $L_E\gg l$, where l is the mean free path. It is thus limited to low energies $E\ll\hbar/\tau$, where τ is the collision time. We believe that $\langle[\delta N(E)]^2\rangle$ stops increasing with E when $E>\hbar/\tau$ because in a perfect crystal, the random potential broadens each level by an amount $\sim\hbar/\tau$. The changes in the number of levels in a band of width $E\gg\hbar/\tau$ are therefore confined to narrow subbands of width $\hbar/\tau$ near the edge of the main band, and these changes are independent of E.

We observe that as the metal-dielectric interface is approached, E_c becomes comparable to the spacing Δ between the levels. Formula (20), and hence also the entire Wigner-Dyson scheme, thus does not apply at all. This is not surprising, since in the dielectric phase (in which all the states are localized) the typical off-diagonal matrix elements H_{ij} are proportional to $\exp(-L/\xi)$ and decay with increasing sample size much faster than the spacing $\Delta\approx L^{-d}$ (here ξ is the localization radius). This fact was noted in Ref. 18 for one-dimensional systems, in which the states are always localized.

3. DERIVATION OF THE PRINCIPAL RESULTS

We will now use the diagram technique[19] to calculate the fluctuations in the density of the energy levels. The density ν_ε of the electron states at a given energy ε in a sample of volume V is

$$\nu_\varepsilon=\frac{s}{V}\sum_i\delta(\varepsilon-\varepsilon_i). \tag{23}$$

Here and below, ν_ε is understood to be the total state density including possible degeneracy (e.g., spin degeneracy) of multiplicitys. As usual, one can express ν_ε in terms of $G_\varepsilon^R(\mathbf{r},\mathbf{r}')$, the exact retarded Green function for an electron of energy ε in the coordinate representation for a specified choice of the impurity potential:

$$\nu_\varepsilon=-\frac{s}{\pi V}\operatorname{Im}\int G_\varepsilon^R(\mathbf{r},\mathbf{r}')\,d\mathbf{r}. \tag{24}$$

After averaging over the different realizations of the random potential, Eqs. (23), (24) yield the familiar expressions for the average density of states ν. However, it is clear already from (23) that the fluctuations in ν_ε must be very large.

The state density correlation function for two different energies ε_1, ε_2 is

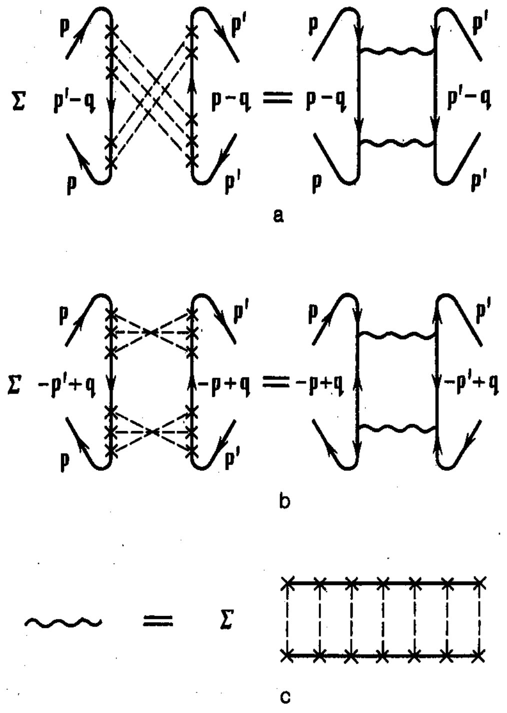

FIG. 3. Diagrams for calculating $K(\varepsilon_1,\varepsilon_2)$: a) diffusion contribution; b) cooperon contribution; c) diffusion (cooperon) pole—a sum over ladder diagrams.

$$K(\varepsilon_1,\varepsilon_2)\equiv\langle\nu_{\varepsilon_1}\nu_{\varepsilon_2}\rangle-\langle\nu_{\varepsilon_1}\rangle\langle\nu_{\varepsilon_2}\rangle$$
$$=\frac{s^2}{\pi^2}\int\frac{d\mathbf{r}_1\,d\mathbf{r}_2}{V^2}\{\langle\operatorname{Im}G_{\varepsilon_1}{}^R(\mathbf{r}_1,\mathbf{r}_1)\operatorname{Im}G_{\varepsilon_2}{}^R(\mathbf{r}_2,\mathbf{r}_2)\rangle$$
$$-\langle\operatorname{Im}G_{\varepsilon_1}{}^R(\mathbf{r}_1,\mathbf{r}_1)\rangle\langle\operatorname{Im}G_{\varepsilon_2}{}^R(\mathbf{r}_2,\mathbf{r}_2)\rangle\}. \tag{25}$$

The correlation function for the electron propagators in the curly brackets can be calculated by the impurity diagram technique. The main contribution comes from the diagrams shown in Fig. 3, which were already considered in Ref. 20, where the spatial correlations of the state density were analyzed. The so-called diffusons (Fig. 3a) and cooperons (Fig. 3b) play a key role in these diagrams. They are obtained by summing ladder diagrams (Fig. 3c) with a small momentum difference or sum, respectively. One of the two electron lines must correspond to the retarded Green function G^R; the other corresponds to the advanced Green function $G^A=(G^R)^*$.

The expression for the correlation function $K(\varepsilon_1,\varepsilon_2)$ corresponding to the diagrams in Fig. 3 can be written as

$$K(\varepsilon_1,\varepsilon_2)=\frac{s^2}{(2\pi^2\nu\tau^2)^2}\left|\int\frac{d\mathbf{p}}{(2\pi)^d}[G_{\varepsilon_1}{}^R(\mathbf{p})G_{\varepsilon_2}{}^A(\mathbf{p})]^2\right|^2$$
$$\times\operatorname{Re}\int\frac{d\mathbf{r}_1\,d\mathbf{r}_2}{V^2}\{[P_{\varepsilon_1-\varepsilon_2}^{(D)}(\mathbf{r}_1,\mathbf{r}_2)]^2+[P_{\varepsilon_1-\varepsilon_2}^{(C)}(\mathbf{r}_1,\mathbf{r}_2)]^2\}$$
$$=\frac{s^2}{2}\operatorname{Re}\int\frac{d\mathbf{r}_1\,d\mathbf{r}_2}{\pi^2V^2}\{[P_{\varepsilon_1-\varepsilon_2}^{(D)}(\mathbf{r}_1,\mathbf{r}_2)]^2+[P_{\varepsilon_1-\varepsilon_2}^{(C)}(\mathbf{r}_1,\mathbf{r}_2)]^2\}, \tag{26}$$

where $P_\omega^{(D)}(\mathbf{r}_1,\mathbf{r}_2)$ and $P_\omega^{(C)}(\mathbf{r}_1,\mathbf{r}_2)$ described a diffuson and a cooperon, respectively, in the coordinate representation and satisfy the equation

$$\{-i\omega-D(\partial_{C,D})^2+\gamma_{C,D}\}P_\omega^{(C,D)}(\mathbf{r}_1,\mathbf{r}_2)=\delta(\mathbf{r}_1-\mathbf{r}_2). \tag{27}$$

Here γ_C and γ_D represent the damping (due, e.g., to inelastic electron collisions), and

$$\partial_D=\frac{\partial}{\partial\mathbf{r}_1},\quad\partial_C=\frac{\partial}{\partial\mathbf{r}_1}+\frac{2ie\mathbf{A}}{c\hbar}, \tag{28}$$

where $\mathbf{A}$ is the vector potential for the external magnetic field $\mathbf{H}$.

For an isolated sample, the boundary conditions for Eq. (27) require that the current across the surface S of the sample must vanish:

$$\mathbf{n}\left.\frac{\partial P_\omega{}^{(C,D)}(\mathbf{r}_1,\mathbf{r}_2)}{\partial\mathbf{r}_1}\right|_{\mathbf{r}\in S}=0, \tag{29}$$

where $\mathbf{n}$ is the normal to S.

Let us assume that $\mathbf{H}=0$ and there is no spin scattering, and that the sample is a parallelelepiped of dimensions L_x, L_y, L_z with $L_x\geqslant L_y\geqslant L_z$. In this case it is easier to solve Eq. (27) in the momentum representation, for which

$$P_\omega^{(D,C)}(\mathbf{q})=\frac{1}{-i\omega+Dq^2+\gamma} \tag{30}$$

and the quantization conditions follow from (29) ($\mu=x,y,z$):

$$q_\mu=\pi n_\mu/L_\mu,\quad n_\mu=0,\ \pm1,\ \pm2\ldots. \tag{31}$$

Expression (26) for $K(\varepsilon_1,\varepsilon_2)$ then becomes

$$K(\varepsilon_1,\varepsilon_2)=-\frac{s^2}{\pi^2V^2}\operatorname{Re}\sum_{n_\mu}\frac{1}{(\varepsilon_1-\varepsilon_2+i\gamma+iDq^2)^2}, \tag{32}$$

where

$$q^2=\pi^2\sum_\mu\frac{n_\mu^2}{L_\mu^2}$$

and we henceforth set $\hbar=1$ in all but the final results. If the sample is so small that $|\varepsilon_1-\varepsilon_2|$ and γ are much less than $E_{c,\mu}=D/L_\mu^2$, then only the term with $n_\mu=0$ must be retained in the summation over n_μ in (32), which thus simplifies to

$$K(\varepsilon_1,\varepsilon_2)=-\frac{s^2}{\pi^2V^2}\operatorname{Re}\frac{1}{(\varepsilon_1-\varepsilon_2+i\gamma)^2}. \tag{33}$$

This shows that the state density correlation function is negative when $|\varepsilon_1-\varepsilon_2|>\gamma$. On the other hand, (33) also shows that $K(\varepsilon_1,\varepsilon_2)$ is positive when $\varepsilon_1=\varepsilon_2$. As $\gamma\to0$ it increases and becomes $\sim\nu^2$ for $\gamma\approx\Delta$. Such large fluctuations are also predicted starting from Eq. (23).

The number of levels (19) in a band of energy width E centered at the point ε is related to the unaveraged density of states (23) by

$$N_\varepsilon(E)=V\int_{\varepsilon-E/2}^{\varepsilon+E/2}\nu_{\varepsilon'}\,d\varepsilon'. \tag{34}$$

According to (25) and (34), the mean square fluctuation in the number of levels is equal to

$$\langle[\delta N(E)]^2\rangle=\int_{\varepsilon-E/2}^{\varepsilon+E/2}d\varepsilon_1\,d\varepsilon_2\,K(\varepsilon_1,\varepsilon_2). \tag{35}$$

Substituting (33) into (35), we find that

$$\langle[\delta N(E)]^2\rangle=\frac{s^2}{\pi^2}\ln\left(\frac{E^2+\gamma^2}{\gamma^2}\right) \tag{36}$$

if $E \ll E_{c,\mu}$. For $E \gg \gamma$ one readily checks that (36) coincides with (20) if $\beta = k = 1$.

Now let $E \gg E_{c,\mu}$ for all μ. In this case energies $|\varepsilon_1 - \varepsilon_2| \gg E_{c,\mu}$ are important in (35), so that the summation over n_μ in (32) can be replaced by an integration over dq_μ. The result for $\gamma \ll E$ is

$$\langle[\delta N(E)]^2\rangle=V\frac{s^2}{\pi^2}\int\frac{d\mathbf{q}}{(2\pi)^3}\ln\left[\frac{E^2}{(Dq^2)^2}+1\right]$$
$$=\frac{2^{1/2}}{6\pi^3}\frac{Vs^2}{L_E^3}=\frac{2^{1/2}s^2}{6\pi^3}\left(\frac{E^3}{E_{c,x}E_{c,y}E_{c,z}}\right)^{1/2}, \tag{37}$$

which for $L_x = L_y = L_z = L$ agrees with Eq. (6) if we set $\beta = k = 1$ and $d = 3$ in the latter.

If L_x, L_y, and L_z differ greatly, we have in addition to the regions $E \gg E_{c,\mu}$ and $E \ll E_{c,\mu}$ considered above the regions $E_{c,x} \ll E \ll E_{c,y}, E_{c,z}$ and $E_{c,x}, E_{c,y} \ll E \ll E_{c,z}$. In the first region the sample is effectively one-dimensional ($d = 1$), while in the second case it is two-dimensional ($d = 2$). To calculate $K(\varepsilon_1,\varepsilon_2)$ and $\langle[\delta N(E)]^2\rangle$ in the first case, one must keep the terms with $n_y = n_z = n$ in (32) and replace the summation over q_x by an integration. In the two-dimensional case, $n_z = 0$ and one must integrate over q_x and q_y. This results in Eq. (6).

We have thus found that the level density fluctuations obey Eq. (20) for small E, while (6) is valid for large E, provided we set $\beta = k = 1$. This result is not surprising—Eqs. (36) and (37) correspond to an orthogonal ensemble because we have so far neglected both the external magnetic field and the spin scattering of the electrons.

To see how the other Dyson ensembles can occur in a metal sample, we consider for simplicity a d-dimensional cube. If a magnetic field H satisfying

$$H \gg H_c=(c\hbar/4De)\max\{(EE_c)^{1/2}, E\} \tag{38}$$

is applied to the system, the cooperon contribution to $K(\varepsilon_1,\varepsilon_2)$ can be neglected. If the Zeeman splitting $g\mu H$ is nonetheless small compared to E, i.e., if $H \ll H_s = E/g\mu$ (we note that in practice one always has $H_c \ll H_s$), then $\langle[\delta N(E)]^2\rangle$ turns out to be just one–half the value given by (36) or (37). The same result also follows from (20) or (6) for a unitary ensemble ($\beta = 2$) with $k = 1$.

In what follows we will need to recall how the diffusons and cooperons depend on the spin variables. It is helpful to picture the diffusion as depending on the total spin j of an electron and hole and on its projection M, while the cooperons depend on the total spin of two electrons and its projection.[21] When $H \gg H_s$, the magnetic field suppresses not only the entire cooperon contribution but also the contribution from diffusons with $M = \pm 1$, so that $K(\varepsilon_1,\varepsilon_2)$ is decreased fourfold. We can use (20) and (6) to interpret this result as follows: Although we are considering the unitary case ($\beta = 2$), in fact we are dealing with two independent series of nondegenerate ($s = 1$, $k = 2$) levels which correspond to different projections of the electron spin.

The symplectic case arises when one considers the spin-orbit scattering of the electrons by ordinary impurities and

$$\hbar/\tau_{so} \gg E, \tag{39}$$

where τ_{s0} is the characteristic spin-orbit scattering time. The scattering suppresses the triplet contributions to the state density correlation function, i.e., the contributions that correspond to unit total spin j (Ref. 21). Only the diffuson and cooperon singlet transitions ($j = 0$) remain important. Therefore, $K(\varepsilon_1,\varepsilon_2)$ is less than the value (33) by a factor of 4. We recall here that the spin-orbit scattering does not violate invariance under time reversal and does not lift the twofold degeneracy of the electron energy levels (the Kramers degeneracy, $s = 2$). The above result thus corresponds to Eqs. (20) and (6) with $\beta = 4$ and $k = 1$, as expected for a symplectic Dyson ensemble. In the intermediate case $E \gg \hbar/\tau_{s0} \gg \gamma$ and $E < E_c$, the mean square level density fluctuation is

$$\langle[\delta N(E)]^2\rangle=\frac{1}{2\pi^2}\left\{\ln\frac{E}{\gamma}+3\ln\frac{E\tau_{s0}}{\hbar}\right\}. \tag{40}$$

If both a magnetic field and spin-orbit scattering are present and conditions (38) and (39) are satisfied, then the state density fluctuations depend only on the singlet diffusion contribution, and $K(\varepsilon_1,\varepsilon_2)$ is one-eighth the value given by (33). The level number fluctuations are thus given by Eqs. (20) and (6) with $\beta = 2$ and $s = k = 1$, i.e., the Dyson ensemble is unitary. Table I lists the coefficients s, k, and β for several values of H and $\hbar/\tau_{s0}$.

TABLE I. Values of the coefficients s, k, and β for several magnetic fields and spin-orbit scattering strengths.

	$H \ll H_c$	$H \ll H \ll H_s$	$H \gg H_s$
$E\tau_{so} \gg \hbar$	$s=2$ $k=1$ $\beta=1$	$s=2$ $k=1$ $\beta=2$	$s=1$ $k=2$ $\beta=2$
$E\tau_{so} \ll \hbar$	$s=2$ $k=1$ $\beta=4$	$s=1$ $k=1$ $\beta=2$	$s=1$ $k=1$ $\beta=2$

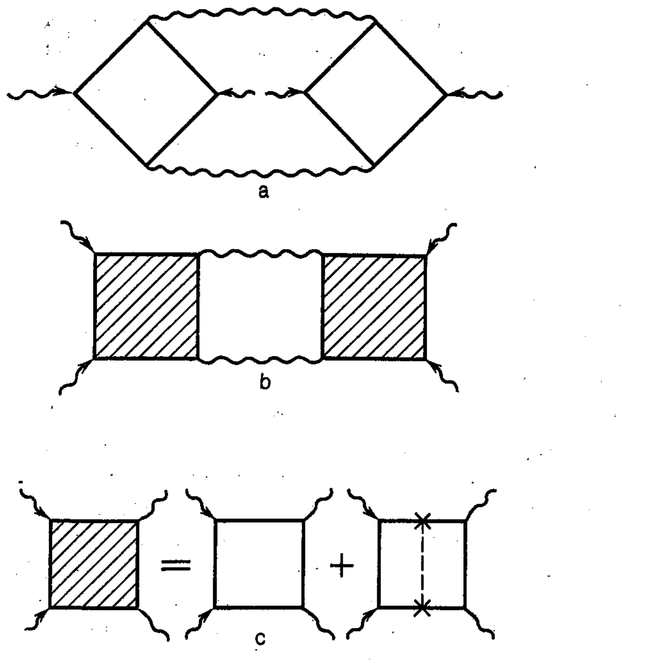

FIG. 4. Diagrams for calculating the correlation function for the fluctuations in the conductivity. The arrows indicate the vector vertices.

We close this section by considering the correlation functions for the number of levels in two different bands of width E centered at energies ε_1 and ε_2. If $E_c \gg E$, $|\varepsilon_1 - \varepsilon_2|$, γ then it follows easily from (33) and (34) that the correlation function for the fluctuations in the level numbers is

$$\langle\delta N_{\varepsilon_1}(E)\,\delta N_{\varepsilon_2}(E)\rangle=\frac{ks^2}{\pi^2\beta}\operatorname{Re}\ln\left[1-\frac{E^2}{(\varepsilon_1-\varepsilon_2+i\gamma)^2}\right], \tag{41}$$

which shows that the correlation is negative if $(\varepsilon_1 - \varepsilon_2)^2 > \gamma^2 + E^2/2$. This implies that the levels must repel one another. For $|\varepsilon_1 - \varepsilon_2| \gg E,\gamma$, the correlation function falls off as the square of the distance between the bands,

$$\langle\delta N_{\varepsilon_1}(E)\,\delta N_{\varepsilon_2}(E)\rangle=-\frac{ks^2}{\pi^2\beta}\frac{E^2}{(\varepsilon_1-\varepsilon_2)^2}. \tag{42}$$

In the region $|\varepsilon_1 - \varepsilon_2| \gg E_c, E$ the magnitude and sign of the correlation function depends on the dimension:

$$\langle\delta N_{\varepsilon_1}(E)\,\delta N_{\varepsilon_2}(E)\rangle=\begin{cases}-\dfrac{2^{1/2}}{8\pi^2}\dfrac{ks^2}{\beta}\dfrac{E^2}{(|\varepsilon_1-\varepsilon_2|^3E_c)^{1/2}}, & d=1\\[2ex] \dfrac{2^{1/2}}{16\pi^3}\dfrac{ks^2}{\beta}\dfrac{E^2}{(E_c^3|\varepsilon_1-\varepsilon_2|)^{1/2}}, & d=3\end{cases} \tag{43}$$

To lowest order in E_c, the correlation vanishes altogether if $d = 2$. All of these results could have been anticipated from Eq. (6), since the proportionality $\langle[\delta N(E)]^2\rangle \propto E^{d/2}$ implies a positive correlation for level numbers in different bands if $d/2 > 1$ and a negative correlation if $d/2 < 1$; finally, it implies that the bands are uncorrelated if $d/2 = 1$.

4. FLUCTUATIONS IN THE CONDUCTIVITY AND DIFFUSION COEFFICIENT

The mean-square fluctuation in the specific static conductivity $\sigma_{\alpha\gamma}$ is given by the sum of the diagrams[1)] in Fig. 4 and has the form

$$\langle\delta\sigma_{\alpha\gamma}\delta\sigma_{\mu\rho}\rangle=\frac{ks^2}{\beta}\left(\frac{\sigma}{\pi\nu V}\right)^2\int\frac{d\varepsilon_1\,d\varepsilon_2}{[4T\operatorname{ch}(\varepsilon_1/2T)\operatorname{ch}(\varepsilon_2/2T)]^2}$$
$$\times\sum_{\mathbf{q}}\{|P^{(D)}_{\varepsilon_1-\varepsilon_2}(\mathbf{q})|^2(\delta_{\alpha\mu}\delta_{\gamma\rho}+\delta_{\gamma\mu}\delta_{\alpha\rho})+\operatorname{Re}[P^{(D)}_{\varepsilon_1-\varepsilon_2}(q)]^2\delta_{\alpha\gamma}\delta_{\mu\rho}\}. \tag{44}$$

We assume that the sample is isotropic on the average and that $\langle\sigma_{\alpha\gamma}\rangle = \sigma\delta_{\alpha\gamma}$. The first and second terms in the curly brackets (44) come from the diagrams in Fig. 4a and 4b, respectively. At zero temperature ($\varepsilon_1 = \varepsilon_2 = 0$)

$$|P^{(D)}_{\varepsilon_1-\varepsilon_2}(\mathbf{q})|^2=\operatorname{Re}[P^{(D)}_{\varepsilon_1-\varepsilon_2}(\mathbf{q})]^2=(Dq^2)^{-2} \tag{45}$$

and straightforward calculations lead to the expression[6–8]

$$\langle\delta G_{\alpha\gamma}\delta G_{\mu\rho}\rangle=(ks^2/\beta)(e^2/\pi^3\hbar)^2b_d\{\delta_{\alpha\mu}\delta_{\gamma\rho}+\delta_{\alpha\rho}\delta_{\gamma\mu}+\delta_{\alpha\gamma}\delta_{\mu\rho}\},$$
$$b_1=\pi^4/90,\quad b_2\approx1{,}51,\quad b_3\approx2.0 \tag{46}$$

for the mean-square fluctuation in the conductance for a d-dimensional cube.

If $\varepsilon_1 \neq \varepsilon_2$, relation (45) breaks down. At a finite temperature, the contributions from the first and second terms in the curly brackets in (44) are thus unequal. The correlation function for the conductance fluctuations is expressible in the form

$$\langle\delta G_{\alpha\gamma}\delta G_{\mu\rho}\rangle=G_D^2\{\delta_{\alpha\mu}\delta_{\gamma\rho}+\delta_{\alpha\rho}\delta_{\gamma\mu}\}+G_\nu^2\delta_{\alpha\gamma}\delta_{\mu\rho}, \tag{47}$$

where the coefficients G_D^2 and G_ν^2 can be calculated from (44). Making the change of variables $\varepsilon_\pm = \varepsilon_2 \pm \varepsilon_2$ and integrating over ε_+ we obtain

$$G_D^2=\frac{ks^2}{\beta}\left(\frac{e^2}{\hbar}\frac{E_c}{\pi}\right)^2\int\frac{d\varepsilon_-}{2T}f\left(\frac{\varepsilon_-}{2T}\right)\sum_q\left|\frac{1}{Dq^2-i\varepsilon_-}\right|^2, \tag{48}$$

$$G_\nu^2=\frac{ks^2}{\beta}\left(\frac{e^2}{\hbar}\frac{E_c}{\pi}\right)^2\int\frac{d\varepsilon_-}{2T}f\left(\frac{\varepsilon_-}{2T}\right)\operatorname{Re}\sum_q\frac{1}{(Dq^2-i\varepsilon_-)^2}, \tag{49}$$

where

$$f(x)=\frac{x\operatorname{cth}x-1}{\operatorname{sh}^2x}. \tag{50}$$

There is good reason for regarding the two terms on the right in (47) as describing independent contributions to $\langle\delta G_{\alpha\gamma}\delta G_{\mu\rho}\rangle$ from the state-density and diffusion-coefficient fluctuations. First, the Einstein relation

$$G_{\alpha\gamma}=e^2L^{d-2}\int\frac{d\varepsilon}{4T\operatorname{ch}^2(\varepsilon/2T)}\nu_\varepsilon D_{\alpha\gamma}(\varepsilon) \tag{51}$$

implies that the state density fluctuations contribute an amount proportional to $\delta_{\alpha\gamma}\delta_{\mu\rho}$. Second, it can be shown that

$$G_\nu^2/\langle G\rangle^2=\langle[\delta\nu(T)]^2\rangle/\nu^2, \tag{52}$$

where

$$\nu(T)=\int\frac{\nu_\varepsilon\,d\varepsilon}{4T\operatorname{ch}^2(\varepsilon/2T)} \tag{53}$$

is the effective density of states at the Fermi level at temperature T. Finally, one can show that the fluctuations $\delta D_{\alpha\gamma}$ and $\delta\nu(T)$ are independent. Indeed, consider the correlation function $\langle\delta\nu(T)\delta G_{\alpha\gamma}\rangle$ defined by the diagram in Fig. 5. A calculation like the one in the preceding section shows that

$$\langle\delta\nu(T)\,\delta G_{\alpha\gamma}\rangle=\frac{\langle G\rangle}{\nu}\langle[\delta\nu(T)]^2\rangle=\frac{\nu G_\nu^2}{\langle G\rangle}, \tag{54}$$

which together with (51) gives

$$\langle\delta\nu(T)\,\delta D_{\alpha\gamma}\rangle=0, \tag{55}$$

i.e., the state-density and diffusion-coefficient fluctuations are independent.

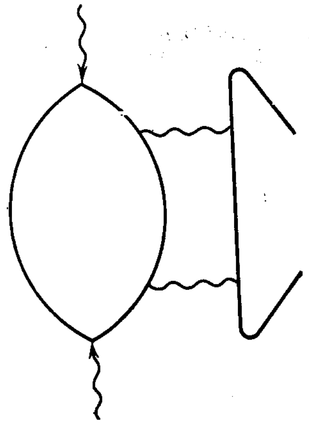

FIG. 5. Diagram for calculating the correlations between the state density and conductivity fluctuations.

Equation (46) implies that the coefficients $G_D{}^2$ and $G_\nu{}^2$ are identical when $T \ll E_c$. If on the other hand $T \gg E_c$ and $d = 3$, we can replace the summation over q in (48) and (49) by an integral to obtain

$$G_\nu{}^2 = \frac{1}{2} G_D{}^2 = \frac{ks^2}{\beta}\left(\frac{e^2}{2\pi\hbar}\right)^2 a_3 \frac{L_T}{L} = \frac{ks^2}{\beta}\left(\frac{e^2}{2\pi\hbar}\right)^2 a_3 \left(\frac{E_c}{T}\right)^{1/2}. \tag{56}$$

Here

$$a_3 = \frac{1}{2\pi}\int_0^\infty \frac{dx}{x^{1/2}} f(x) = \frac{3}{2(2\pi)^{1/2}}\zeta\left(\frac{1}{2}\right) \approx 0.874, \tag{57}$$

and $\zeta(x)$ is the Riemann zeta-function. It follows readily from Eq. (53) and the preceding discussion that for $T \gg E_c$ and a sample of arbitrary dimension d

$$G_\nu{}^2 = (ks^2/\beta)(e^2/2\pi\hbar)^2 a_d (E_c/T)^{(4-d)/2}. \tag{58}$$

The a_d can be calculated from (49):

$$a_d = \frac{2\pi(d-6)}{(2\pi^{1/2})^d \sin(\pi d/4)}\zeta\left(\frac{d}{2}-1\right), \quad a_2 = 1, \quad a_1 \approx 2.61.$$

The relation $G_D \approx G_\nu$ breaks down for small d, because in contrast to (49), the low-energy region $\varepsilon_- \ll T$ is important in the integration over ε_- in (48). The result for $d = 2$ is

$$G_D{}^2 = \frac{ks^2}{\beta}\left(\frac{e^2}{2\pi\hbar}\right)^2 \frac{E_c}{T} \ln\frac{T}{\max\{E_c, \hbar/\tau_\varphi\}} \gg G_\nu{}^2, \tag{59}$$

where $\hbar/\tau = \gamma$ is the decay of the electron energy levels due to inelastic collisions.

The difference between G_D and G_ν is even greater in the one-dimensional case:

$$G_D{}^2 = \frac{ks^2}{12\pi\beta}\left(\frac{e^2}{\hbar}\right)^2 \frac{E_c}{T}\varphi\left(\frac{E_c\tau_\varphi}{\hbar}\right) = \frac{ks^2}{12\pi\beta}(e^2/\hbar)^2 \begin{cases} E_c/3T, & E_c\tau_\varphi \gg \hbar \\ E_c{}^{3/2}/T(\hbar/\tau_\varphi)^{1/2}, & E_c\tau_\varphi \ll \hbar \end{cases}, \tag{60}$$

where

$$\varphi(x) = x^{1/2}\,\mathrm{cth}\,x^{-1/2} - x.$$

Equation (47), (56), (59), and (60) lead to expressions for the conductance fluctuations as functions of T for $d = 1,2,3$; these expressions were derived in Ref. 8.

The amplitude of the periodic and aperiodic oscillations in the conductance of a thin metal sample observed in Ref. 2 decreases as $T^{-1/2}$ with increasing temperature. Such a dependence $\delta G(T)$ also follows from (60) and (47), since the experiments in Ref. 2 dealt with the situation $d = 1$ and $E_c\tau_\phi > \hbar$. On the other hand, Eq. (49) predicts a $T^{-3/4}$ dependence for the fluctuations $\delta\nu(T)$ in the density of states under these conditions.

The reason for this marked difference in the fluctuation magnitudes for the diffusion coefficient and state density is that the latter fluctuations are suppressed by the repulsion among the levels. We have already observed in discussing Eq. (33) that this repulsion causes the correlation function $K(\varepsilon_1,\varepsilon_2)$ to change sign near the singularity at small $\varepsilon_1 - \varepsilon_2$, so that these energies do not contribute to the integrals in (35) and (49). On the other hand, the low-energy region is important in the integral over ε_- in (48) for low dimensions $d \leq 2$. This reflects the fact that unlike the state density correlation function, the correlation function for the diffusion coefficients at unequal energies is always positive.

We are grateful to A. G. Aronov, B. Z. Spivak, and D. E. Khmel'Nitskiĭ for helpful discussions, and to L. N. Bulaevskiĭ and M. V. Sadovskiĭ for providing us with a preprint of their results.

[1)] The diagrams in Fig. 4b were left out in Refs. 6, 8, and 9, and we are grateful to A. G. Aronov for bringing them to our attention. We also note that if the Matsubara method is used to calculate the correlation functions, or if only results at $T = 0$ are needed,[19] one need consider only those diagrams that have field vertices at which the sign of the electron energy changes (see the Appendix to Ref. 22).

---

[1] R. A. Webb, S. Washburn, C. P. Umbach, and R. B. Laibowitz, Phys. Rev. Lett. **54**, 485 (1985).

[2] S. Washburn, C. P. Umbach, R. B. Laibowitz, and R. A. Webb, Phys. Rev. **32**, 4789 (1985).

[3] V. Chandrasekhar, M. J. Rooks, S. Wind, and D. E. Prober, Phys. Rev. Lett. **55**, 1610 (1985).

[4] M. Buttiker, Y. Imry, and M. Ya. Azbel, Phys. Rev. **A30**, 1982 (1984); M. Buttiker, Y. Imry, R. Landauer, and S. Pinhas, Phys. Rev. **B31**, 6207 (1985).

[5] A. D. Stone, Phys. Rev. Lett. **54**, 2692 (1985).

[6] P. A. Lee and A. D. Stone, Phys. Rev. Lett. **55**, 1622 (1985).

[7] B. L. Al'tshuler, Pis'ma Zh. Eksp. Teor. Fiz. **41**, 530 (1985) [JETP Lett. **41**, 648 (1985)].

[8] B. L. Al'tshuler and D. E. Khmel'nitskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 291 (1985) [JETP Lett. **42**, 359 (1985)].

[9] B. L. Al'tshuler and B. Z. Spivak, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 363 (1985) [JETP Lett. **42**, 447 (1985)].

[10] D. J. Thouless, Phys. Rev. Lett. **39**, 1167 (1977).

[11] E. P. Wigner, Ann. Math. **53**, 36 (1951); **62**, 548 (1955); **65**, 203 (1957); **67**, 325 (1958).

[12] C. E. Porter ed., Statistical Theories of Spectra: Fluctuations, Academic Press, New York (1965).

[13] T. A. Brody, J. Flores, J. B. French *et al.*, Rev. Mod. Phys. **53**, 385 (1981).

[14] L. P. Gor'kov and G. M. Éliashberg, Zh. Eksp. Teor. Fiz. **43**, 1407 (1965).

[15] K. B. Efetov, Zh. Eksp. Teor. Fiz. **83**, 833 (1982) [Sov. Phys. JETP **56**, 467 (1982)].

[16] F. J. Dyson, J. Math. Phys. **3**, 140, 157, 166 (1962).

[17] F. J. Dyson and M. L. Mehta, J. Math. Phys. **4**, 701 (1963).

[18] K. B. Efetov and A. I. Larkin, Zh. Eksp. Teor. Fiz. **85**, 764 (1983) [Sov. Phys. JETP **58**, 444 (1983)].

[19] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, *Methods of Quantum Field Theory in Statistical Physics*, Prentice-Hall, N. J. (1963).

[20] L. N. Bulaevskiĭ and M. V. Sadovskiĭ, Pis'ma Zh. Eksp. Teor. Fiz. **43**, 76 (1986).

[21] B. L. Altshuler and A. G. Aronov, in: Electron-Electron Interactions in Disordered Systems (A. L. Efros and M. Pollak eds.), North Holland (1985).

[22] B. L. Al'tshuler, A. G. Aronov, A. I. Larkin, and D. E. Khmel'nitskiĭ, Zh. Eksp. Teor. Fiz. **81**, 768 (1981) [Sov. Phys. JETP **54**, 411 (1981)].

Translated by A. Mason