T = 1°K. Just as in the case of the Fe—Sn system, a small change of the gap, ~5%, is observed only at the largest concentrations. A reduction of the R(V) curves, similar to that performed earlier for the Fe—Sn system, is shown in Figs. 5b and 5c. As seen from Fig. 5b, a linear dependence of ln(1 − f) on N is observed also for the Mn—Sn system at Mn concentrations larger than 1.5×10^{15} at/cm^2. Up to these concentrations, the presence of Mn does not affect the tunnel characteristics. We attribute the presence of such a region (<1.5×10^{15} at/cm^2) to the systematic error in the determination of the Mn concentration from the readings of the quartz balance. Since the Mn was evaporated from a tantalum vessel, it was possible for the evaporator to become contaminated by the volatile impurities sublimated before the Mn, which produced a shift in the determination of the concentration. The value of r_S calculated for Mn is 2.8 ± 0.5 Å.

The dependence of ln(1 − f) on N, obtained for the Cr—Sn system, is much more complicated (Fig. 5c) and is not described by a simple linear relation. But the dimension of the normal region around the impurity atom, at low impurity concentrations, can be obtained from the value of df/dn as N → 0. Estimates made in this manner yield for Cr a value r_S = 3.6 ± 0.5 Å.

The appearance of an inflection on the ln[1 − f(N)] curve (Fig. 5c) can be attributed to at least two causes: 1) structural peculiarities in the formation of the Cr film, 2) decrease of the magnetic moment of the Cr atoms at large Cr concentrations on the barrier boundary.

It should be noted, however, that the arguments presented above for the Fe—Sn system are fully applicable also to Cr. When the Cr was evaporated, the substrate temperature was lower than the recrystallization temperature, and when the Cr film is produced the critical dimension of the nucleus is also of the order of the dimension of the atom. Consequently, the structure of the Cr layer should differ little from the Fe layer, and the observed difference seems to be due to a decrease of the magnetic moment of the Cr atoms.

It was shown in a number of studies[14–17] that the presence of a magnetic impurity in a superconductor leads to a spatial change in the order parameter. In this paper we approximate the spatial dependence of the order parameter by means of a simple step function

$$\Delta(r)=\begin{cases}0, & r<r_*\\ \Delta_0, & r>r_*\end{cases},$$

and this appears to be a good approximation in the case of the Fe—Sn system. For the Mn—Sn and Cr—Sn system, possibly, one should use a more complicated function Δ(r). This yields, naturally different values of f(N) for the maxima and minima of the R(V) curves.

For Cr concentrations larger than 6×10^{14} at/cm^2, a discrepancy is observed between the values of f(N) obtained from different points of the curves. The values of f(N) obtained from the maximum of the R(V) curves at T = 1.6°K are approximately 20% lower than those obtained from the minima of the curves obtained at T ~ 1°K, although an appreciable section (up to 6×10^{14} at/cm^2) and the overall course of the curves coincide. The obtained picture agrees qualitatively with that given in[15]. A detailed comparison is hardly meaningful, since Heinricks's theory[15] is confined to the region T ~ T_c and makes no allowance for the Kondo effect and for the interaction between the impurities, which probably plays an important role in the investigated systems.

We observed in the studied systems in the normal state no attributes of giant anomalies at V = 0, which were attributed in a number of papers to electron scattering by magnetic impurities[18,19]. Thus, in the voltage region <10 mV of interest to us, the value of ΔR(V)/R(0) for the systems Cr—Sn and Mn—Sn did not exceed 8% at N < 3×10^{15} at/cm^2.

The junction resistance of the Fe—Sn system was independent of the voltage.

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Calculation of critical indices for zero-gap semiconductors

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The indices defining the exponents in the power laws in the Green functions and physical characteristics of zero-gap semiconductors of the second kind in the strong-coupling region (i.e., the region of energies $\omega \ll \omega_0$, where $\omega_0 = m^* e^4/2\kappa_0^2$ is the exciton binding energy, and momenta $k \ll k_0$, where $k_0 = m^* e^2/\kappa_0$ is the inverse Bohr radius of the exction) are calculated. Two methods are applied: expansion in $\epsilon = 4 - d$ (d is the dimensionality of space) and expansion in the "large number of components," i.e., in $1/\Pi$, where Π is the polarization operator. In the first approximation the two procedures give results that are numerically slightly different but qualitatively similar. It follows from the results that there are several energy and momentum regions: (1) the "free region," (2) the region $\omega_1 < \omega < \omega_0$, $k_1 < k < k_0$ ($\omega_1 \sim 2.5 \times 10^{-4}\ \omega_0$, $k_1 \sim 1.6 \times 10^{-2} k_0$), in which the "random-phase approximation" is applicable, i.e., the interaction of the carriers is altered but their spectrum is conserved, and (3) the true strong-coupling region $\omega < \omega_1$, $k < k_1$. In the regions (2) and (3) the results of Abrikosov and Beneslavskiĭ are applicable, with $\alpha = \nu = 2$ in region (2) and $\alpha = 1.76$, $\nu = 1.92$ in region (3).

1. INTRODUCTION

It has been shown earlier (cf.[1,2]) that in zero-gap semiconductors of the second kind (e.g., α-Sn), a strong-coupling regime arises in the vicinity of the point of contact of the branches of the spectrum (a point of the order of $k_0 = m^* e^2/\kappa_0$ along the momentum axis and $\omega_0 = m^* e^4/2\kappa_0^2$ along the energy axis, where m^* is the effective mass and κ_0 is the dielectric permittivity). In this region the Green functions and vertices acquire the self-similar form:

$$G \sim \omega_0^{-1} \left(\frac{k}{k_0}\right)^{-\alpha} g\left[\frac{\omega/\omega_0}{(k/k_0)^\nu}\right],$$
$$\Gamma \sim \frac{\omega_0}{k_0^2} \left(\frac{k}{k_0}\right)^{\nu-3} d\left[\frac{\omega/\omega_0}{(k/k_0)^\nu}\right], \qquad (1)$$
$$\gamma \sim \left(\frac{k}{k_0}\right)^{\alpha-\nu} q\left[\frac{\omega/\omega_0}{(k/k_0)^\nu}\right].$$

The indices α and ν determine the behavior of physical quantities in the singular region, e.g., their dependence on the temperature, magnetic field, frequency, etc.

In view of the fact that the problem of zero-gap semiconductors is methodologically similar to the theory of phase transitions, to determine the unknown indices α and ν it is natural to attempt to apply the methods developed in recent years by Wilson and Fisher[3] and Ferrell and Scalapino[4] and successfully used for the analysis of different types of phase transitions. The present article is devoted to this question.

2. CONTINUATION TO NONINTEGER DIMENSIONS

The calculation of the indices should be performed for a specific model. Although the indices do not depend on the interaction constant, they can depend on the geometry (the little group and specific representation; cf.[1]). We shall consider a very realistic situation, namely, the case of α-Sn with a diamond-type lattice. The bands touch at a point with little group O_h. We take a representation corresponding to $J = {}^3/_2$; the corresponding Hamiltonian and spectrum are well known[1]. To simplify matters, we put $3\beta = \gamma$ in the formula in[1]; the spectrum then possesses spherical symmetry. The Hamiltonian in this case can be written in the form

$$H = fk^2 + c[\hat{J}_\mu \hat{J}_\nu - {}^1/_3 \hat{J}^2 \delta_{\mu\nu}] k_\mu k_\nu, \qquad (2)$$

whence it can be seen that there is invariance with respect to any rotations. The eigenvalues are doubly degenerate and equal to

$$\varepsilon_{1,2} = (f \pm c) k^2. \qquad (3)$$

Hence it can be seen that a zero-gap spectrum is obtained for c > f.

It was shown in[1] that the zeroth-approximation term for the polarization operator is proportional to an integral that converges like 1/k at the lower limit (k is the external momentum). It is clear that in four-dimensional space we would have a logarithmic integral. This corresponds precisely to the situation in phase transitions. The application of Wilson's method[3] is associated with formulation of the problem in a space with a noninteger number of dimensions $d = 4 - \epsilon$. In the present case, the problem is complicated by the fact that operators J_μ associated with the three-dimensional rotation group appear in the Hamiltonian and form tensor combinations with the momentum components. The continuation to noninteger dimensions in such a situation has been effected for the case $J = 1$[5], but the method used is not applicable for $J = {}^3/_2$. In view of this, we shall proceed in the following way.

We write the second term in the Hamiltonian (2) in the form cH_1,

$$H_1 = k_\mu k_\nu \hat{A}_{\mu\nu}, \qquad (4)$$

and analyze the properties of the operators $A_{\mu\nu}$. According to (2), these operators possess the following properties:

$$\sum_\mu \hat{A}_{\mu\mu} = 0, \quad \{A_{\mu\mu}, A_{\nu\nu}\} = -1 \ (\nu \neq \mu),$$
$$\mathrm{Sp}\, \hat{A}_{\mu\nu} = 0, \quad A_{\mu\mu}^2 = 1, \quad \hat{A}_{\mu\nu}^2 = {}^3/_4 \ (\nu \neq \mu); \qquad (5)$$

$\{\hat{A}_{\mu\nu}, \hat{A}_{\rho\sigma}\} = 0$ in all other cases. Here there is no summation over repeated indices. All these properties can be written in the form of a single formula:

$$\{\hat{A}_{\mu\nu}, \hat{A}_{\rho\sigma}\} = {}^3/_2(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}) - \delta_{\mu\nu}\delta_{\rho\sigma}. \qquad (6)$$

Hence it follows that the anticommutator of two Hamiltonians H_1 at different points k is equal to

$$\{H_1(\mathbf{k}_1), H_1(\mathbf{k}_2)\} = 3(\mathbf{k}_1\mathbf{k}_2)^2 - k_1^2 k_2^2. \qquad (7)$$

In particular, if $k_1 = k_2 = k$ we obtain $2c^2k^4$, and averaging (7) over the directions of k_2 gives zero. These properties are carried over in an obvious way to a space with noninteger dimensions. For this it is sufficient to require that

$$\{H_1(k_1),\ H_1(k_2)\} = \frac{2}{d-1}[d(k_1k_2)^2 - k_1^2k_2^2]. \tag{8}$$

For this, we can take H_1 in the form (4), where the operator coefficients $\hat{A}_{\mu\nu}$ satisfy the condition

$$\{\hat{A}_{\mu\nu},\ \hat{A}_{\rho\sigma}\} = \frac{d}{d-1}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}) - \frac{2}{d-1}\delta_{\mu\nu}\delta_{\rho\sigma}. \tag{9}$$

In the following we shall need to use the property (8) more frequently than any other. However, in the calculation of the polarization operator it is necessary to take the trace, and we need to know the rank of the matrices $\hat{A}_{\mu\nu}$. In a space of three dimensions this is equal to 4. We shall determine the rank from the condition that it be sufficient to enable the conditions (9) to be satisfied. We shall use the following method. For the three-dimensional case, the matrices $A_{\mu\nu}$ can be represented with the aid of direct products of two sets of Pauli matrices:

$$\begin{aligned}
&A_{zz} = J_z^2 - \frac{5}{4} = \sigma_z s_z, \quad A_{xy} = \frac{1}{2}\{J_x, J_y\} = \frac{\sqrt{3}}{2} s_y,\\
&A_{xx} = J_x^2 - \frac{5}{4} = \frac{\sqrt{3}}{2} s_x - \frac{1}{2}\sigma_z s_z, \quad A_{xz} = \frac{1}{2}\{J_x, J_z\} = \frac{\sqrt{3}}{2} s_z\sigma_x,\\
&A_{yy} = J_y^2 - \frac{5}{4} = -\frac{\sqrt{3}}{2} s_x - \frac{1}{2}\sigma_z s_z, \quad A_{yz} = \frac{1}{2}\{J_y, J_z\} = \frac{\sqrt{3}}{2} s_z\sigma_y.
\end{aligned} \tag{10}$$

A similar representation can also be used in other cases. The operators $A_{\mu\nu}$ are symmetric in μ and ν, and $\Sigma A_{\mu\mu} = 0$. If the dimensionality of space (i.e., the number of values of μ) is d, then the number of such matrices equals $(d-1)(d+2)/2$.

The conditions (9) can be satisfied by means of linear combinations of matrices with zero trace, if these matrices anticommute, their squares are equal to unity, and there are $(d-1)(d+2)/2$ of them. This is proved as follows. We have at our disposal $(d-1)(d+2)/2$ matrices, which we denote by $\gamma_1, \gamma_2, \ldots$. We take any $(d-1)d/2$ of them as the $A_{\mu\nu}$ with $\mu \neq \nu$. There remain $d-1$ matrices, in terms of which we must express the d matrices $A_{\mu\mu}$. We must then have: a) $A_{\mu\mu}^2 = 1$, b) $\{A_{\mu\mu}, A_{\nu\nu}\} = -2/(d-1)$ $(\mu \neq \nu)$, c) $\sum_\mu A_{\mu\mu} = 0$. We denote $A_{11} \equiv \gamma_1$. We choose $A_{22} = p\gamma_1 + q\gamma_2$ in such a way that the conditions (a) and (b) are satisfied. This gives $p = -1/(d-1)$, $q = [1 - 1/(d-1)^2]^{1/2}$. We next choose $A_{33} = p_1\gamma_1 + q_1\gamma_2 + r\gamma_3$, again to satisfy the same conditions. This gives

$$p_1 = -\frac{1}{d-1}, \quad q_1 = -\frac{1}{d-2}\left[1 - \frac{1}{(d-1)^2}\right]^{1/2},$$
$$r = \left[1 - \frac{1}{(d-1)^2}\right]^{1/2}\left[1 - \frac{1}{(d-2)^2}\right]^{1/2}.$$

Continuing in the same spirit, we obtain expressions for all the $A_{\mu\mu}$ (for integer d):

$$\begin{aligned}
A_{11} &= \gamma_1,\\
A_{22} &= -\frac{1}{d-1}\gamma_1 + \left[1 - \frac{1}{(d-1)^2}\right]^{1/2}\gamma_2,\\
A_{33} &= -\frac{1}{d-1}\gamma_1 - \frac{1}{d-2}\left[1 - \frac{1}{(d-1)^2}\right]^{1/2}\gamma_2 +\\
&+ \left[1 - \frac{1}{(d-1)^2}\right]^{1/2}\left[1 - \frac{1}{(d-2)^2}\right]^{1/2}\gamma_3.\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

$$A_{d-1\,d-1} = -\frac{1}{d-1}\gamma_1 - \ldots + \left[1 - \frac{1}{(d-1)^2}\right]^{1/2}\cdots\left[1 - \frac{1}{(d-d+2)^2}\right]^{1/2}\gamma_{d-1},$$
$$A_{dd} = -\frac{1}{d-1}\gamma_1 - \ldots - \left[1 - \frac{1}{(d-1)^2}\right]^{1/2}\cdots\left[\frac{3}{4}\right]^{1/2}\gamma_{d-1}.$$

The condition (c) is fulfilled automatically. This is not surprising, as it already follows from (9) that $\{A_{\rho\sigma}, \sum_\nu A_{\nu\nu}\} = 0$ for any of the matrices $A_{\rho\sigma}$.

If we construct such matrices as direct products of Pauli matrices, we obtain the following cases:

for n = 1,

$$\sigma_x,\ \sigma_y,\ \sigma_z;$$

for n = 2,

$$s_x,\ s_y,\ s_z\sigma_x,\ s_z\sigma_y,\ s_z\sigma_z;$$

for n = 3,

$$\nu_x,\ \nu_y,\ \nu_z s_x,\ \nu_z s_y,\ \nu_z s_z\sigma_x,\ \nu_z s_z\sigma_y,\ \nu_z s_z\sigma_z.$$

Thus, in a similar manner, we find that the number of matrices equals $3 + 2(n-1)$, where n is the number of types of Pauli matrices. This number should be equal to (or greater than) $(d-1)(d+2)/2$. The rank of the matrices equals 2^n. Hence we obtain

$$r(d) = 2^n = 2^{d(d+1)/4-1}. \tag{11}$$

In particular, for d = 3 we have r = 4; if d = 4, r = 16. The rule (8) and the rank (11) of the matrices are sufficient in practice for our purposes.

3. FIRST APPROXIMATION IN $\epsilon = 4 - d$

According to the philosophy of Wilson's approach[3], we can argue in the following way. We shall assume that $d = 4 - \epsilon$. If we put $\epsilon = 0$, we obtain logarithmic integrals. This gives us the possibility of choosing principal diagrams in which the power of the interaction constant corresponds to the power of a large logarithm. As a result of summation, certain functions of this logarithm arise. Then, assuming ϵ to be small, we carry out the formal replacement:

$$\ln\frac{k_0}{k} = \frac{1}{2\pi^2}\int_k^{k_0}\frac{d^4k}{k^4} \to \text{const}\int_k^{k_0}\frac{d^{4-\epsilon}k}{k^4} = \text{const}\,\frac{k^{-\epsilon} - k_0^{-\epsilon}}{\epsilon}.$$

For $k \ll k_0$ we are left with $k^{-\epsilon}/\epsilon$. Putting $\epsilon = 1$, we obtain the critical exponents for all the quantities.

We shall carry out the above program for our model. We put the external frequency equal to zero. Then according to[1], $G \sim \omega_0^{-1}(k/k_0)^{-\alpha}$. First of all we shall consider the polarization operator (Fig. 1). In the first approximation, this is equal to

$$\Pi^{(1)}(\omega, k) = -i\,\mathrm{Sp}\int G^{(0)}\left(\omega_1 - \frac{\omega}{2},\ k_1 - \frac{k}{2}\right) \times G^{(0)}\left(\omega_1 + \frac{\omega}{2},\ k_1 + \frac{k}{2}\right)\frac{d^4k_1}{(2\pi)^4}\frac{d\omega}{2\pi}. \tag{12}$$

We put d = 4. According to the arguments expressed in[1], the integral over ω_1 goes to zero for k = 0. Consequently, we must expand in k under the integral. We then obtain an integral of the type $k^2\int dk_1/k_1$. The situation turns out to be very similar to that which was found in an zero-gap semiconductors of the first kind (with a linear spectrum). This is confirmed by the fact that an estimate of the correction to γ represented in Fig. 2 gives only unimportant terms. In view of this, we can confine ourselves to summing the logarithmic terms in the electron Green function G and Coulomb-quantum Green function Γ, taking $\gamma = 1$.

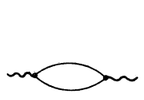

FIG. 1

FIG. 2

We call attention to the following circumstance. The Hamiltonian $fk^2 + cH_1$ is a linear combination of two invariants of the rotation group. According to[1], we must assume that in the logarithmic approximation the constants f and c are replaced by functions of $\ln(k_0/k)$. However, it is not difficult to see that logarithmic corrections arise only in the term with H_1. In fact, assuming that $\omega \ll ck^2$, we have for the self-energy in the zeroth approximation

$$\Sigma^{(1)}(0, k) = i\frac{4\pi e^2}{\varkappa_0}\int\frac{1}{k_1^2}G^{(0)}(k + k_1, \omega_1)\frac{d\omega_1}{2\pi}\frac{d^4k_1}{(2\pi)^4}. \tag{13}$$

The function $G^{(0)}(k_1)$ can be written in the form

$$G^{(0)}(k_1) = (\omega_1 - fk_1^2 - cH_1)^{-1} = \frac{\omega_1 - fk_1^2 + cH_1}{(\omega_1 - fk_1^2)^2 - (ck_1^2 - i\delta)^2}. \tag{14}$$

Substituting this expression into (13) and taking the integral over ω_1, we find

$$\Sigma^{(1)}(0, k) = \frac{2\pi e^2}{\varkappa_0}\int\frac{1}{k_1^2}\frac{H_1(k + k_1)}{(k + k_1)^2}\frac{d^4k_1}{(2\pi)^4}. \tag{15}$$

But it immediately follows from this that $\mathrm{Tr}\,\Sigma = 0$ (since $\mathrm{Tr}\,H_1 = 0$). Consequently, a logarithmic correction to the first invariant fk^2 does not arise, and only the correction to H_1 remains. This means that the diagonal term in G^{-1} for $\omega \ll ck^2$ has the power α_1, equal to $2 - O(\epsilon^2)$. But if the term of the type H_1 has a lower power (of the type $\alpha = 2 - O(\epsilon)$), and below we shall verify that this is precisely the case, then in the approximation under consideration it is sufficient to retain only this term. In view of this, we shall assume below that f = 0.

Earlier[1], the general case of a logarithmic situation with one invariant in the Hamiltonian was analyzed. The result reduces to the following. It is necessary to determine the quantities a and b from the relations

$$a\ln\frac{k_0}{k} = -\Gamma^{(0)}\Pi^{(1)}, \quad b\ln\frac{k_0}{k} = \frac{\{\Sigma^{(1)}(k), H_1(k)\}}{2ck^4}. \tag{16}$$

In the region where the logarithms are large,

$$G(0, k) \sim G^{(0)}\left(\ln\frac{k_0}{k}\right)^{-b/(a+b)}, \quad \Gamma(0, k) \sim \Gamma^{(0)}\left(\ln\frac{k_0}{k}\right)^{-a/(a+b)}. \tag{17}$$

Replacing $\ln(k_0/k) \to k^{-1}$, we obtain

$$\alpha = 2 - \frac{b}{a+b}, \quad \nu - 3 = -2 + \frac{a}{a+b}, \tag{18}$$

i.e., $\alpha = \nu$, which, incidentally, was obvious, since $\gamma \sim (k/k_0)^{\alpha-\nu} = 1$ in the approximation under consideration. ν can differ from α in second order in ϵ. But in the first approximation, according to (18), it is necessary to determine the ratio b/a, and from it

$$\alpha = \nu = 2 - \frac{b/a}{1 + b/a}. \tag{18a}$$

According to (12), (14) (with f = 0) and (11), we have

$$\Pi^{(1)}(\omega, k) = -ir(d)\int\{\omega_+\omega_- + c^2[d(k_+k_-)^2 - k_+^2k_-^2]/(d-1)\} \times[(\omega_- - ck_-^2 + i\delta)(\omega_- + ck_-^2 - i\delta)(\omega_+ - ck_+^2 + i\delta)(\omega_+ + ck_+^2 - i\delta)]^{-1}\frac{d\omega}{2\pi}\frac{d^dk}{(2\pi)^d},$$

where $k_\pm = k_1 \pm k/2$ and $\omega_\pm = \omega_1 \pm \omega/2$. Integrating over the frequencies, we obtain

$$\Pi^{(1)}(\omega, k) = -\frac{r(d)}{2c}\frac{d}{d-1} \times\int\frac{(k_+^2 + k_-^2)[k_+^2k_-^2 - (k_+k_-)^2]d^dk_1}{[(k_+^2 + k_-^2 - i\delta) - (\omega/c)^2]k_+^2k_-^2(2\pi)^d}. \tag{19}$$

We shall put $\omega = 0$ and confine ourselves to logarithmic accuracy. Assuming that $k \ll k_1$, we find

$$\Pi^{(1)}(0, k) = -\frac{r(d)}{4c}\frac{d}{d-1}\int\frac{k_1^2k^2 - (k_1k)^2}{k_1^6}\frac{d^dk_1}{(2\pi)^d}. \tag{20}$$

We put d = 4; in the integral,

$$\frac{d^dk_1}{(2\pi)^d} \to 4\pi\int_0^\pi\sin^2\theta\,d\theta\int\frac{k_1^3dk_1}{(2\pi)^4}.$$

We then obtain

$$a = \frac{1}{c}\frac{4\pi\cdot 2\pi^2}{(2\pi)^4}\frac{e^2}{\varkappa_0}4. \tag{21}$$

We turn now to the self-energy. It follows from symmetry considerations that $\Sigma^{(1)} = qcH_1$, where q can depend only on k^2. Taking the anticommutator with $H_1(k)$, according to (9) we obtain

$$2qk^4 = \frac{4\pi e^2d}{c\varkappa_0(d-1)}\int\frac{1}{k_1^2}\left\{\frac{(k_1k + k^2)^2}{(k_1 + k)^2} - \frac{(k_1k)^2}{k_1^2}\right\}\frac{d^dk_1}{(2\pi)^d}$$

(here we have made use of the relation $\overline{(k_1\cdot k)^2} = k_1^2k^2/d$). Assuming that $k \ll k_1$ and expanding in k, we obtain

$$q = \frac{2\pi e^2d}{c\varkappa_0(d-1)}\int(1 + 4\cos^4\theta - 5\cos^2\theta)\frac{d^dk_1}{(2\pi)^dk_1^4}.$$

Since $q = b\ln(k_0/k)$ (cf. 17), we obtain from this (d = 4)

$$b = \frac{1}{c}\frac{4\pi\cdot 2\pi^2}{(2\pi)^4}\frac{e^2}{\varkappa_0}\frac{1}{6}. \tag{22}$$

Substituting this expression into (18), we have

$$\alpha = \nu = 2 - 1/25. \tag{23}$$

It is interesting to note that the indices α and ν for the electron G-function differ little from the "free indices." As regards the function $\Gamma(k)$, it becomes close to the quantity $\sim k^{-1}$ obtained from the "random-phase approximation," i.e., obtained on the basis of a calculation of the first-approximation diagram for the polarization operator[6].

Furthermore, in the approximation under consideration it turns out that the term with fk^2 in the Hamiltonian is not renormalized, and $\alpha = \nu$. The next approximations would undoubtedly introduce corrections to these statements. However, it is extremely difficult to calculate the corresponding integrals and we shall apply another approximate method, which, moreover, gives an idea of how the transition to the strong-coupling regime occurs.

4. THE APPROXIMATION OF A LARGE NUMBER OF COMPONENTS

In the preceding section, it was noted that indices are obtained that are close to those given by taking into account only the first approximation for the polarization operator. This circumstance is characteristic of the situation in phase transitions when the "order parameter" has a large number of components. Although this case is rare in true physical models, we can nevertheless count on the fact that expansion in the number of components gives a series that is not too slowly convergent even for n = 3. Such an approximation was proposed by Ferrell and Scalapino[4]. The special role of the first-order diagram for the polarization operator is

explained by the fact that, as a consequence of taking the trace, it is proportional to n. Since $\Gamma \sim 1/\pi$ appears in the self-energy Σ and, at the same time, there is no summation over the components, Σ will be of order $1/n$. In our case, the number of components is equal to four and we can count on the good convergence of this procedure.

Although, as we shall see below, the $1/n$ expansion leads in first order to more complicated calculations than does the ϵ expansion, it has important merits. First of all, the calculations are performed in the real three-dimensional space and questions associated with the nonuniqueness of the continuation to noninteger d do not arise. It is then found that, even in the first approximation, $\alpha \neq \nu$. Having a real model, we shall perform the calculation only for this model, and not for arbitrary n.

The expression for the first approximation for the polarization operator is obtained from (19), if we put $d = 3$. Introducing the variables $x = \cos\theta$, $z = 2k_1/k$ and $s = 2\omega/ck^2$, we obtain

$$\Pi^{(1)} = -\frac{3k}{2c\pi^2}\int_0^1 dx \int_0^\infty dz \frac{z^4(z^2+1)(1-x^2)\,dx}{[(z^2+1-is\delta)^2 - s^2][(z^2+1)^2 - 4z^2x^2]}. \tag{24}$$

Integrating over x, we obtain

$$\Pi^{(1)} = -\frac{3k}{8c\pi^2}\int_0^\infty \frac{z^2 dz}{(z^2+1-is\delta)^2 - s^2}\left[z^2+1-\frac{1}{2}\frac{(z^2-1)^2}{z}\ln\frac{z+1}{|z-1|}\right]. \tag{25}$$

The integral over z can be taken if we note that

$$\int_0^\infty dz \to \frac{1}{2}\int_{-\infty+i\delta}^{\infty+i\delta} dz,$$

with

$$\ln\frac{z+1}{|z-1|} \to \ln\frac{z+1}{z-1}.$$

We then obtain

$$\Pi^{(1)} = -\frac{3k}{32c\pi}\varphi(s), \tag{26}$$

$$\varphi(s) = \frac{(2+s)^2}{s}\operatorname{arctg}(1+s)^{-1/2} - \frac{(2-s)^2}{s}\operatorname{arctg}(1-s)^{-1/2} - (1+s)^{1/2} - (1-s)^{1/2} \tag{27a}$$

for $0 < s < 1$, and

$$\varphi(s) = \frac{(2+s)^2}{s}\operatorname{arctg}(1+s)^{-1/2} - (1+s)^{1/2} - \frac{\pi(2-s)^2}{2s}\theta(2-s) + i\left[(s-1)^{1/2} - \frac{(2-s)^2}{2s}\ln\left|\frac{1+(s-1)^{1/2}}{(s-1)^{1/2}-1}\right|\right] \tag{27b}$$

for $s > 1$;

$$\varphi(-s) = \varphi(s); \tag{27c}$$

the asymptotic values are

$$\varphi(s) \to 4\left(\frac{\pi}{2} - 1\right), \quad s \to 0; \qquad \varphi(s) \to \frac{8}{3\sqrt{s}}(1+i), \quad s \to \infty; \tag{28}$$

the singular values are

$$\varphi(1) = 9\operatorname{arctg}\frac{1}{\sqrt{2}} - \frac{\pi}{2} - \sqrt{2}, \qquad \varphi(2) = \frac{4\pi}{3} - \sqrt{3} + i. \tag{29}$$

Graphs of the real and imaginary parts of the function φ ($\varphi = \varphi_1 + i\varphi_2$) are given in Fig. 3.

It will be important for the following that $\varphi(s)$ is the value on the real axis of a function of the complex variable s, defined by the formula (27a), analytic in the plane of the complex variable s with cuts along the real axis for $1 < |s| < \infty$, and symmetric with respect to $s \to -s$. The formulas (27b) and (27c) correspond to taking the value on the upper side along the right cut $1 < s < \infty$ and on the lower side along the left cut $-\infty < s < -1$.

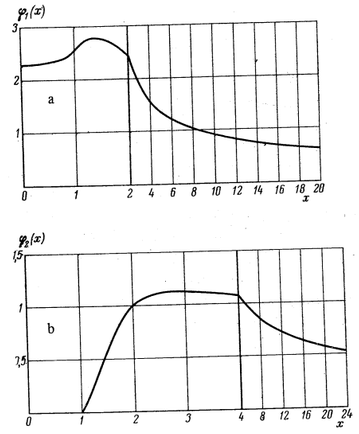

FIG. 3

On decrease of k and ω in the approximation under consideration, we enter first of all a region in which $\Pi^{(1)} \gg \Gamma^{(0)}$. However, the self-energy correction to the electron Green function in this region is still small, and only on further decrease of ω and k does it begin to play a fundamental role. By assumption (cf.[1]), it is precisely then that there arise the power dependences $G \sim \omega^{-\alpha/\nu}$ for $\omega/\omega_0 \gg (k/k_0)^\nu$ or $G \sim k^{-\alpha}$ for $\omega/\omega_0 \ll (k/k_0)^\nu$ (we recall that $\alpha < \nu$ and $\alpha < 2$). Matching between these laws and the free Green function should occur in the region in which Σ is still small. This is possible only in the case when α/ν is close to 1 and α is close to 2.

In fact, for $\omega/\omega_0 \gg (k/k_0)^\nu$,

$$G^{-1} \propto \omega^{\alpha/\nu} \propto \omega\left(\frac{\omega}{\omega_0}\right)^{-(1-\alpha/\nu)} \approx \omega\left[1 + \left(1 - \frac{\alpha}{\nu}\right)\ln\frac{\omega_0}{\omega}\right].$$

On the other hand,

$$G^{-1} = \omega - cH_1 - \Sigma.$$

Hence it follows that in the matching region, for $\omega/\omega_0 \gg (k/k_0)^\nu$,

$$\Sigma(\omega) \approx -\left(1 - \frac{\alpha}{\nu}\right)\omega\ln\frac{\omega_0}{\omega}. \tag{30}$$

In exactly the same way, if we assume that $\Sigma = qcH_1$ for $\omega/\omega_0 \ll (k/k_0)^\nu$, we have

$$G^{-1} \propto k^\alpha \propto k^2\left(\frac{k}{k_0}\right)^{-(2-\alpha)} \approx k^2\left[1 + (2-\alpha)\ln\frac{k_0}{k}\right]$$

and

$$q = (2-\alpha)\ln(k_c/k). \tag{31}$$

Thus in the approximation under consideration, determination of Σ with logarithmic accuracy is sufficient to determine α and ν. Of course, all these arguments are valid only in the case when $1 - \alpha/\nu \ll 1$ and $2 - \alpha \ll 1$. The self-energy is obtained from (13) by replacing $\Gamma^{(0)} = 4\pi e^2/\kappa_0 k^2$ by $\Gamma = -1/\pi$. In addition, it is necessary to take into account that $\Sigma(0, 0)$ must be subtracted from $\Sigma(\omega, k)$ (renormalization of the chemical potential). For finite ω we have

$$\Sigma(\omega, k) = -i\int \Pi^{-1}(\omega_1, k_1) G(\omega - \omega_1, k - k_1)\, d\omega_1 \frac{d^3k_1}{(2\pi)^4}. \tag{32}$$

The function $G^{(0)}$ can be written in the form

$$G^{(0)}(\omega - \omega_1, k - k_1) = \frac{1}{2}\left[1 + \frac{H_1(k-k_1)}{\varepsilon_1(k-k_1)}\right][\omega - \omega_1 - \varepsilon_1(k-k_1) + i\delta]^{-1} + \frac{1}{2}\left[1 - \frac{H_1(k-k_1)}{\varepsilon_1(k-k_1)}\right][\omega - \omega_1 + \varepsilon_1(k-k_1) - i\delta]^{-1}, \tag{33}$$

where $\epsilon_1(k) = ck^2$.

For Π we substitute the expression (26), (27). Owing to the fact that this is the value on the real axis of an analytic function with two cuts (with an appropriate definition on the cut), the integral over ω_1 can be represented as a contour integral, where the contour C is depicted in Fig. 4a, in which the poles from $G^{(0)}(\omega - \omega_1)$ are also shown. Since the integral over ω_1 for $\Sigma(\omega, k) - \Sigma(0, 0)$ converges in any case, the contour can be closed and deformed in such a way that it takes the form C′ (Fig. 4b). Since the function has complex-conjugate values on the two sides of the cut, we obtain (we do not write the integral to be subtracted for $\Sigma(0, 0)$)

$$\Sigma = \frac{8}{3\pi^2}\int_1^\infty ds_1 \int ck_1\, dk_1 \frac{d\Omega}{4\pi} A(s_1) \times \left\{\left[1 + \frac{H_1(k-k_1)}{\varepsilon_1(k-k_1)}\right]\left[\frac{2\omega}{ck_1^2} - s_1 - \frac{2(k-k_1)^2}{k_1^2}\right]^{-1} + \left[1 - \frac{H_1(k-k_1)}{\varepsilon_1(k-k_1)}\right]\left[\frac{2\omega}{ck_1^2} - s_1 + \frac{2(k-k_1)^2}{k_1^2}\right]^{-1}\right\} + \frac{8}{3\pi}\int dk_1 ck_1 \frac{d\Omega}{4\pi} B(s_{10})\left[\frac{H_1(k-k_1)}{\varepsilon_1(k-k_1)} - 1\right], \tag{34}$$

where $s_1 = 2\omega/ck_1^2$, $s_{10} = 2\omega/ck_1^2 + 2(k - k_1)^2/k_1^2$ is the value of s_1 at the pole, $A(s_1) = -\mathrm{Im}[1/\varphi(s_1)]$, and $B(s_1) = \mathrm{Re}[1/\varphi(s_1)]$ (we assume that $k \ll k_1$ and $2\omega \ll ck_1^2$; this is valid in the calculation of the logarithmic part of the integrals—cf. below). This expression can be transformed by making use of properties of the function $1/\varphi(s) = B(s) - iA(s)$, namely, the analytic properties and the symmetry in s. Making use of the Cauchy theorem for the contour C″ (Fig. 4c), we obtain

$$B(s_{10}) = -\frac{1}{\pi}\int_1^\infty A(s_1)\left[\frac{1}{s_1 + s_{10}} + \frac{1}{s_1 - s_{10}}\right] ds_1. \tag{35}$$

Substituting this relation into (37), we obtain

$$\Sigma = \frac{8}{3\pi^2}\int_1^\infty ds_1 \int ck_1\, dk_1 \frac{d\Omega}{4\pi} A(s_1) \cdot \times \left\{\left[1 + \frac{H_1(k-k_1)}{\varepsilon_1(k-k_1)}\right]\left[\frac{2\omega}{ck_1^2} - s_1 - \frac{2(k-k_1)^2}{k_1^2}\right]^{-1} + \left[1 - \frac{H_1(k-k_1)}{\varepsilon_1(k-k_1)}\right]\left[\frac{2\omega}{ck_1^2} + s_1 + \frac{2(k-k_1)^2}{k_1^2}\right]\right\}. \tag{36}$$

In this form we need no longer imply the subtraction of $\Sigma(0, 0)$. In fact, in view of the fact that $\int d\Omega H_1(k_1) = 0$, the quantity $\Sigma(0, 0) = 0$.

If $\omega \gg ck^2$ we can put $k = 0$. Expanding in ω and integrating over k_1, we obtain

$$\Sigma(\omega) = -\frac{16}{3\pi^2}\int_1^\infty \frac{A(s)}{(s+2)^2} ds\, \omega \ln\frac{\omega_0}{\omega}, \tag{37}$$

where $\omega_0 \sim ck_0^2$; k_0 is the upper limit of the integration over k. But if $\omega \ll ck^2$, we must, on the contrary, put $\omega = 0$ in the integral (36). We then obtain

$$\Sigma(k) = -\frac{16}{3\pi^2}\int_1^\infty ds_1 \int ck_1\, dk_1 \frac{d\Omega}{4\pi} A(s_1) \frac{H_1(k-k_1)}{\varepsilon_1(k-k_1)} \frac{1}{s_1 + 2(k-k_1)^2/k_1^2}.$$

Since its trace equals zero, this expression should be proportional to H_1; $\Sigma = qcH_1$. Calculating the anticommutator with H_1 and making use of the rule (7), we have

$$q = -\frac{8}{3\pi^2 k^4}\int_1^\infty ds_1 \int k_1\, dk_1 \frac{d\Omega}{4\pi} A(s_1) \times \frac{3[k \times (k-k_1)]^2 - k^2(k-k_1)^2}{(k-k_1)^2} \frac{1}{s + 2(k-k_1)^2/k_1^2}$$

Expanding in k/k_1, we obtain

$$q = -\frac{8}{3\pi^2}\left[\frac{2}{5}\int_1^\infty \frac{A(s)}{s+2} ds - \frac{16}{5}\int_1^\infty \frac{A(s)}{(s+2)^2} ds + \frac{64}{15}\int_1^\infty \frac{A(s)}{(s+2)^3} ds\right].$$

However, this expression is not completely accurate. This can be seen from the fact that the first term in it diverges as $s \to \infty$. Indeed, according to (28), $A(s) \propto s^{1/2}$ as $s \to \infty$. The inaccuracy has arisen in the use of the transformation (35), which is valid only for the case when A(s) and B(s) fall off at infinity[1)]. We can correct the situation by subtracting from $B - iA$ a symmetric function of s with the same cuts and asymptotic value, in such a way that the result now be a decreasing function. We note that the trouble arises entirely from the terms of second order in k/k_1, which originate from H_1/ϵ_1 (the zeroth approximation is taken from s_{10} and the denominators of the first term in (34)). Therefore, we shall consider only these terms.

Subtracting from $1/\varphi(s)$ a function which we denote by $1/\varphi_\infty(s) = B_\infty - iA_\infty$, we obtain

$$B(2) = B_\infty(2) - \frac{1}{\pi}\int_1^\infty [A(s) - A_\infty(s)]\left(\frac{1}{s+2} + \frac{1}{s-2}\right) ds.$$

As a result, it turns out that in the expression for q we must make the replacement

$$\int_1^\infty \frac{A(s)}{s+2} ds \to -\frac{\pi}{2}B_\infty(2) + \int_1^\infty \left[\frac{A(s)}{s+2} - A_\infty(s)\frac{s}{s^2-4}\right] ds.$$

For $1/\varphi_\infty(s)$ we can take the simplest function:

$$1/\varphi_\infty(s) = {}^3/_{16}[(s+1)^{1/2} + (1-s)^{1/2}];$$

then $B_\infty = (3/16)(s+1)^{1/2}$ and $A_\infty = ({}^3/_{16})(s-1)^{1/2}$. Substituting $1/\varphi_\infty$ into the integrals, we find

$$\int_1^\infty \frac{A(s)}{s+2} ds \to -\frac{3\sqrt{3}\pi}{32} + \int_1^\infty \left(\frac{A(s)}{s+2} - \frac{3}{16}\frac{s(s-1)^{1/2}}{s^2-4}\right) ds.$$

But we can do still better. We add and subtract $3/16\sqrt{s}$ under the integral. We then obtain

$$\int_1^\infty \frac{A(s)}{s+2} ds \to -\frac{3}{8} + \int_1^\infty \left(\frac{A(s)}{s+2} - \frac{3}{16\sqrt{s}}\right) ds.$$

Carrying out these replacements in q and comparing the expressions for q and for $\Sigma(\omega)$ with (31) and (30), we obtain

$$2 - \alpha = \frac{2}{5\pi^2} - \frac{16}{15\pi^2}\left(A_1 - 8A_2 + \frac{32}{3}A_3\right), \tag{38}$$

$$1 - \frac{\alpha}{\nu} = \frac{16}{3\pi^2}A_2, \tag{39}$$

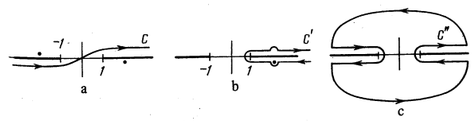

FIG. 4

$$A_1=\int_1^\infty\left(\frac{A(s)}{s+2}-\frac{3}{16\sqrt{s}}\right)ds,\quad A_{n>1}=\int_1^\infty\frac{A(s)}{(s+2)^n}ds. \tag{40}$$

The function $A(s) = -\mathrm{Im}[1/\varphi(s)] = \mathrm{Im}\,\varphi(s)/|\varphi(s)|^2$, and $\varphi(s)$ is given by the formula (27b). Numerical calculation gives:

$$A_1=-0.6976;\quad A_2=0.1570;\quad A_3=0.01207;\quad A_4=0.00183.$$

As a result,

$$2-\alpha=0.238,\quad 1-\alpha/\nu=0.0847 \tag{41a}$$

or

$$\alpha=1.76,\quad \nu=1.92. \tag{41b}$$

The result obtained is evidence that the approach used is highly successful. In fact, both $1-\alpha/\nu$ and $(2-\alpha)/2$ are of the order of 0.1. In view of this, the values obtained for α and ν can be regarded as fairly reliable. But if we take the first approximation of the ϵ expansion, it turns out there that the electron Green function is weakly renormalized, but, in return, the Coulomb-quantum Green function is strongly renormalized: $\delta = 3-\nu \approx 1$, in place of 2 as in Coulomb's law. This shows that, in the given problem, $\epsilon = 4 - d$ is not as suitable an expansion parameter as $1/n$, in contrast to the theory of phase transitions. This is also indicated by the fact that, according to Sec. 3, $1-\alpha/\nu$ is a quantity of order ϵ^2, whereas $2-\alpha$ is of order ϵ. In reality, as can be seen from (41a), they are of the same order (nevertheless, it is true that there is a certain tendency for $1-\alpha/\nu$ to be small). In view of this, we must conclude that, although the ϵ-expansion method can be applied in the given problem for certain qualitative estimates (the absence of the term f, and arguments in favor of the $1/n$ expansion; cf. also Sec. 5), for quantitative calculations it is necessary to use the $1/n$ expansion.

In the derivation performed we have assumed the Hamiltonian to be equal to cH_1, on the basis of the result from the preceding Section that the contribution fk^2 in the Hamiltonian is not renormalized. Actually, this is somewhat inconsistent. In fact, this result was obtained in the first approximation of the ϵ expansion, whereas here we are using a completely different approximation. Nevertheless, a certain argument can be adduced in favor of the validity of the procedure used. Suppose that $f \neq 0$. In this case, the replacement

$$s\to s-2\eta zx,$$

where $\eta = f/c$, takes place in the expression (24) for the polarization operator. Let η be small. In this case, expanding in η gives a correction of order η^2 to Π. Substituting Π into Σ leads to the appearance of corrections of order η^2 in all terms. In particular, a term proportional to $\eta^2 k^2 \ln(k_0/k)$ can arise in Σ. Combining this with the term $-fk^2$ in $G^{(0)-1}$, we arrive at the conclusion that for small η the power $2-\alpha$ in the isotropic term of the Hamiltonian will be proportional to η, i.e., it will be considerably smaller than the constant obtained for the term of the type cH_1. But this means that, in this case, we can neglect the isotropic term in the asymptotic region $\omega, k \to 0$. Consequently, the case we have considered is internally consistent. It is not difficult to see that the opposite assumption, namely, that $f \neq 0$, $c = 0$, leads to $\Pi^{(1)} = 0$, i.e., to the absence of a strong-coupling situation. Consequently, the possibility that the power of the isotropic term is lower than the power of the term of the type cH_1 cannot be realized. Finally, equality of the powers is also impossible, since, if this were the case, it would be true for all f.

One can also discuss all this in a different way. Matching of the power laws with perturbation theory is not always possible. In particular, in matching with ordinary perturbation theory in the ϵ-expansion method[3] it is necessary to select the value of the interaction constant for which this is possible. In our case, for $ck^2 \gg \omega$, there can be terms of the type H_1 and an isotropic term in G^{-1}. We can make three assumptions: a) the power of the term of the type H_1 is lower than the power of the isotropic term. Then matching with the perturbation theory starting from $f = 0$ should exist, and has been effected in this Section; b) the power of the term of the type H_1 is higher than the power of the isotropic term. In this case, there should be matching of the strong-coupling regime with the perturbation theory starting from $c = 0$, and this is not obtained; c) finally, by assuming the powers to be equal we can take any value of the ratio $f/c = \eta$ in the initial Hamiltonian. The two powers should turn out to be equal for any η. But this is not corroborated for small η.

Thus, the possibility (a) considered above is the only one.

5. "ISOTROPIZATION" IN THE SINGULAR REGION

Up to now we have considered only the spherically symmetric model with the Hamiltonian (2). It was found that the power of the second term, which we called cH_1, in the singular region is lower than the power of the term fk^2. This corresponds to the fact that the electron and hole masses are, as it were, equalized in the singular region[2]. We now show that "isotropization" of the spectrum occurs in the singular region, i.e., even in the case when we do not assume isotropy at large momenta, it arises automatically in the singular region.

For this we note that an anisotropic model corresponds to replacement of cH_1, the part of the Hamiltonian given by formula (4), by $c_1H_1 + c_2H_2$, where H_1 includes the terms with the "non-diagonal" matrices $A_{\mu\nu}$ ($\mu \neq \nu$) and H_2 includes the terms with the matrices $A_{\nu\nu}$. We call attention to the fact that, according to the rules (5) and (6), we can easily replace the normalization of the matrices $A_{\mu\nu}$ with $\mu \neq \nu$ without damaging the whole scheme. This gives the possibility of generalizing the method proposed in Sec. 1 for continuing to noninteger dimensions for the case under consideration, and for all integer d the number of matrices γ_i and, consequently, their rank are not increased in comparison with the isotropic model.

Without repeating all the arguments, we cite the principal formulas. Writing

$$c_1H_1+c_2H_2\equiv c_2\sum_{\eta\nu}A_{\eta\nu}k_\eta k_\nu, \tag{42}$$

we have in place of (9)

$$\{A_{\eta\nu},A_{\rho\sigma}\}=\frac{d\mu^2}{d-1}(\delta_{\eta\rho}\delta_{\nu\sigma}+\delta_{\eta\sigma}\delta_{\nu\rho})-\frac{2}{d-1}\delta_{\eta\nu}\delta_{\rho\sigma}+\frac{2d}{d-1}(1-\mu^2)\delta_{\eta\nu\rho\sigma}, \tag{43}$$

where $\mu = c_1/c_2$, and $\delta_{\mu\nu\rho\sigma} = 1$ when all the indices coincide and zero otherwise. It follows from (43) that

$$\{c_1H_1(k_1)+c_2H_2(k_1),c_1H_1(k_2)+c_2H_2(k_2)\}=\frac{2c_2^2}{d-1}\Big[d\mu^2(k_1k_2)^2-k_1^2k_2^2+d(1-\mu^2)\sum_\nu k_{1\nu}^2k_{2\nu}^2\Big]. \tag{44}$$

In particular, for $k_1 = k_2 = k$ the right-hand side becomes equal to $2(\epsilon/c_2)^2$;

$$\varepsilon=c_2k^2\left[\mu^2+\frac{2d}{d-1}(\mu^2-1)A_{22}\right]^{1/2} \tag{45}$$

is the pole of the bare electron Green function;

$$A_{22}=\sum_{\mu\neq\nu}n_\mu^2n_\nu^2.$$

With the same rule (11) for the rank of the matrices, we can solve the problem. To obtain the first approximation in $\epsilon = 4 - d$ it is sufficient to perform the calculations with logarithmic accuracy for $d = 4$. Since in the present case we have not one but two invariants, the scheme we constructed earlier[1] is not directly applicable. However, the basic line of the calculations is the same. We shall find Π and Σ in the logarithmic approximation, assuming c_1 and c_2 (or c_2 and μ) to be functions of $\xi = \ln(k_0/k)$, and then obtain equations from which it will be possible to determine them.

Substituting $G = [\omega - c_1H_1 - c_2H_2]^{-1}$ into (12), taking the integral over ω and transforming the expression obtained, we find

$$\Pi=-4k^2\frac{2\pi^2}{(2\pi)^4}\int_0^\xi\frac{d\xi_1}{c_2(\xi_1)}\left\{\frac{\mu^2+{}^4\!/_9(\mu^2-1)(4\mu^2-1)A_{22}-{}^8\!/_3(\mu^2-1)^2A_{222}}{[1+{}^8\!/_3(\mu^2-1)A_{22}]^{3/2}}\right\}_{av}, \tag{46}$$

where

$$\{\}_{av}=(2\pi^3)^{-1}\int_0^\pi\sin^2\theta\,d\theta\int_0^\pi\sin\varphi\,d\varphi\int_0^{2\pi}d\gamma\{\},$$

A_{22} was defined earlier,

$$A_{222}=\sum_{\mu\neq\nu\neq\rho}n_\mu^2n_\nu^2n_\rho^2,$$

and the projections n_ν of the unit vector are equal to:

$$n_1=\cos\theta,\quad n_2=\sin\theta\cos\varphi,\quad n_3=\sin\theta\sin\varphi\cos\gamma,$$
$$n_4=\sin\theta\sin\varphi\sin\gamma.$$

In (46), μ depends on ξ_1. It follows from (46) that

$$\Gamma=\frac{4\pi e^2}{\varkappa_0(k^2-4\pi e^2\Pi/\varkappa_0)}\equiv\frac{4\pi e^2}{\varkappa_0k^2}d(\xi), \tag{47}$$

where $d(\xi)$ is a logarithmic function.

We turn now to the self-energy. We write, analogously to formula (15),

$$\Sigma^{(1)}(0,\mathbf{k})=\frac{2\pi e^2}{\varkappa_0}\int\frac{d(\xi_1)}{(\mathbf{k}_1+\mathbf{k})^2}\frac{c_1H_1(\mathbf{k}_1)+c_2H_2(\mathbf{k}_1)}{\varepsilon(\mathbf{k}_1)}\frac{d^dk_1}{(2\pi)^d}$$
$$\approx\frac{8\pi e^2}{\varkappa_0}\int\frac{d(\xi_1)(\mathbf{k}_1\mathbf{k})}{k_1^6}\frac{c_1H_1(\mathbf{k}_1)+c_2H_2(\mathbf{k}_1)}{\varepsilon(\mathbf{k}_1)}\frac{d^dk_1}{(2\pi)^d}=q_1H_1(\mathbf{k})+q_2H_2(\mathbf{k}). \tag{48}$$

We take the anticommutator with $A_{\mu\nu}$ ($\mu \neq \nu$). Comparing the two parts and applying the definition $c_1 = c_{10} + q_1$, we obtain the equation

$$c_1(\xi)=c_{10}+\frac{2\pi^2}{(2\pi)^4}\frac{4\pi e^2}{\varkappa_0}{}^2\!/_3\cdot\int_0^\xi\left\{\frac{d(\xi_1)\mu(\xi_1)A_{22}}{(1+{}^8\!/_3A_{22}[\mu^2(\xi_1)-1])^{1/2}}\right\}_{av}. \tag{49}$$

By taking the anticommutator of the formula (48) with $A_{\rho\rho}$, we have, in the same way,

$$c_2(\xi)=c_{20}+\frac{2\pi^2}{(2\pi)^4}\frac{4\pi e^2}{\varkappa_0}2\int_0^\xi\left\{\frac{d(\xi_1)({}^1\!/_4-{}^2\!/_3A_{22})}{(1+{}^8\!/_3A_{22}[\mu^2(\xi_1)-1])^{1/2}}\right\}_{av}. \tag{50}$$

From (46) and (47) we have

$$\frac{1}{d(\xi)}=1+\frac{4\pi e^2}{\varkappa_0}\frac{2\pi^2}{(2\pi)^4}\cdot$$

$$\times4\int_0^\xi\left\{\frac{\mu^2+{}^4\!/_9(\mu^2-1)(4\mu^2-1)A_{22}-{}^8\!/_3(\mu^2-1)^2A_{222}}{[1+{}^8\!/_3(\mu^2-1)A_{22}]^{3/2}}\right\}_{av}d\xi_1. \tag{51}$$

From Eqs. (49), (50) and (51) we can find the three functions c_1, c_2, and d. Our purpose is not this, however, but the determination of the laws

$$c_1\sim\xi^{\alpha_1},\quad c_2\sim\xi^{\alpha_2},\quad d\sim\xi^{-\alpha_3} \tag{52}$$

in the asymptotic region. Using the notation $\beta = 2\times2\pi^2\times4\pi e^2/(2\pi)^4\varkappa_0$, introducing $\mu = c_1/c_2$ and differentiating the equations with respect to ξ, we have

$$\frac{d\ln\mu}{d\xi}=\frac{\beta d(\xi)}{c_2(\xi)}\left\{\frac{A_{22}-{}^1\!/_4}{[1+{}^8\!/_3A_{22}(\mu^2-1)]^{1/2}}\right\}_{av}, \tag{53a}$$

$$\frac{d\ln c_2}{d\xi}=\frac{\beta d(\xi)}{c_2(\xi)}\left\{\frac{{}^1\!/_4-{}^2\!/_3A_{22}}{[1+{}^8\!/_3A_{22}(\mu^2-1)]^{1/2}}\right\}_{av}, \tag{53b}$$

$$\frac{d\ln d^{-1}}{d\xi}=\frac{\beta d(\xi)}{c_2}\left\{\frac{\mu^2+{}^4\!/_9(\mu^2-1)(4\mu^2-1)A_{22}-{}^8\!/_3(\mu^2-1)^2A_{222}}{[1+{}^8\!/_3A_{22}(\mu^2-1)]^{3/2}}\right\}_{av}. \tag{53c}$$

First of all, we make the assumption that μ = const. For this it is necessary that the right-hand side of (53a) vanish, or, in other words,

$$\frac{1}{2\pi^2}\int_0^\pi\sin^2\theta\,d\theta\int_0^\pi\sin\varphi\,d\varphi\int_0^{2\pi}d\gamma\frac{A_{22}-{}^1\!/_4}{[1+{}^8\!/_3A_{22}(\mu^2-1)]^{1/2}}=0. \tag{54}$$

This equation is satisfied for $\mu = 1$, since $\{A_{22}\}_{av} = {}^1\!/_4$. This solution corresponds to the isotropic case. But in principle (54) could also have other solutions.

We note that the left-hand side of Eq. (54) has the asymptotic forms[3)]

$$\left\{\frac{A_{22}-{}^1\!/_4}{(1-{}^8\!/_3A_{22})^{1/2}}\right\}_{av},\quad \mu\to0, \tag{55a}$$

$$\frac{1}{|\mu|}\left\{\frac{A_{22}-{}^1\!/_4}{({}^8\!/_3A_{22})^{1/2}}\right\}_{av},\quad \mu\to\infty. \tag{55b}$$

The maximum value of A_{22}, corresponding to $n_1^2 = n_2^2 = n_3^2 = n_4^2 = {}^1\!/_4$, equals ${}^3\!/_8$, and the minimum value equals zero. Since $\overline{A_{22}} = {}^1\!/_4$, we should expect that values $A_{22} > {}^1\!/_4$ will play the principal role in the averaging of (55a), i.e., the result will be positive, and that values $A_{22} < {}^1\!/_4$ will play the main role in averaging of (55b), i.e., the result will be negative.

These arguments are reinforced by the following estimate. We expand the integrand in (54) in $A_{22} - {}^1\!/_4$ and confine ourselves to the first nonvanishing term. The integral is then easily taken and we obtain

$$-\frac{4}{3}\left\{\left(A_{22}-\frac{1}{4}\right)^2\right\}_{av}\frac{(\mu^2-1)}{[1+{}^2\!/_3(\mu^2-1)]^{3/2}}=-\frac{1}{120}\frac{(\mu^2-1)}{[1+{}^2\!/_3(\mu^2-1)]^{1/2}}. \tag{56}$$

This formula is accurate near $\mu^2 = 1$ and somewhat less accurate at the limits (the expansion parameter is of the order of ${}^1\!/_3$). But it is, however, sufficient for the following conclusions to be drawn: 1) there is only one value $\mu^2 = 1$ at which the expression (54) vanishes; 2) for $\mu \to 0$ this expression is positive and for $\mu \to \infty$ the coefficient of $|\mu|^{-1}$ is negative.

We return now to Eqs. (53) and make an assumption about the powers in (52): $\alpha_1 > \alpha_2$. This means that for $\xi \to \infty$ we have $\mu \to \infty$, i.e., the power of μ is positive. But for this the right-hand side of (53a) must be positive for $\mu \to \infty$ (d cannot become negative and in reality c_2 implies $|c_2|$). But it is proportional to (55b), and, according to the analysis carried out above, the coefficient of $1/|\mu|$ is, in reality, negative. The same contradiction arises if we assume that the powers in (52) satisfy $\alpha_1 < \alpha_2$, i.e., $\mu \to 0$ as $\xi \to \infty$.

Thus, it follows from the arguments given that μ^2

= 1 is the only solution, i.e., "isotropization" arises in the asymptotic region, irrespective of the initial conditions (c_{10} and c_{20}).

6. CONCLUSION

Although the calculations performed above pertained to the singular region, in which all quantities are strongly renormalized because of the interaction, nevertheless, since we have used a matching method, we can analyze qualitatively the transition to this region.

Deviation from the "free" behavior begins for $k < k_0$, $\omega < \omega_0$. So long as the momenta and energies are not too small (we shall denote the lower boundary by k_1, ω_1), the change in the electron Green function can be neglected. In the region $k_1 < k < k_0$, $\omega_1 < \omega < \omega_0$, which we shall call the second region (the first is the "free" region), only the electron-electron interaction describable by the function Γ changes. The inclusion of the zeroth-approximation polarization operator has already been made by a number of authors[6]. Our calculation justifies this approximation in the second region. The Coulomb law in this region is replaced by an interaction proportional to $1/r^2$ (in the coordinates). For $\omega \gg ck^2$, allowance for the polarization leads to the appearance of a dielectric permittivity proportional to $1/\omega^{1/2}$. From formulas (26) and (28) it is not difficult to obtain

$$\varkappa = \varkappa_0 + \frac{e^2(1+i\,\mathrm{sign}\,\omega)}{(2\omega c)^{1/2}}. \qquad (57)$$

In this form, this formula is also true for $\omega \gtrsim \omega_0$. It can be shown that formula (57) also remains valid if we take an isotropic model, but with different masses, i.e., $f \neq 0$. In this case, $c = {}^1/_4(1/m_e + 1/m_h)$.

Since the "free" Green function and the interaction have the form (1) with $\alpha = \nu = 2$, all the temperature and other dependences found earlier in[2] are operative in the second region. As regards estimating the coefficients, the effect can be appreciably reduced in those cases in which interaction of the electrons is important. For example, the electron-electron scattering probability will be proportional to the number $(1 - \alpha/\nu)$ (41a), which is approximately 0.08. In view of the fact that the second region is very simple for calculations, all the expressions for the physical effects in this region can be found, without great complications, in an explicit form with numerical coefficients (in any case, for the isotropic model)[4)], and this has been done in a number of papers (cf. the literature cited in[7]).

According to Sec. 4, change of the electron Green function begins either when

$$k \lesssim k_1 \sim k_0 e^{1/(\alpha-2)} \sim 1{,}6 \cdot 10^{-2} k_0,$$

or when

$$\omega \lesssim \omega_0 e^{1/(\alpha/\nu-1)} \sim 10^{-5}\omega_0.$$

These limits are somewhat different. In fact, $ck_1^2 \sim 2.5 \times 10^{-4}\omega_0$, whence it follows that, in principle, there is a certain region of energies in which Σ is substantially renormalized and, at the same time, the dependence of Σ on ω can be neglected. But this means in practice that a single-particle spectrum remains for energies $2.5 \times 10^{-4}\,\omega_0 > \omega > 10^{-5}\,\omega_0$, although it differs from the "free" spectrum. For $\omega < 10^{-5}\,\omega_0$, the single-particle spectrum is strongly damped.

However, all these changes occur in the region of extremely low energies. We shall call this the third region. In the known zero-gap semiconductors, because of the large dielectric permittivity $\varkappa_0$ and the small effective mass, the second region is itself shifted into the region of energies of the order of $10^{-3} - 10^{-4}$ eV, so that even this region, in the purest samples, may be difficult to "seek out." In principle, we can assume that substances with significantly larger values of ω_0 will be found. However, even in a case in which ω_0 reaches 10 eV, the third region will remain extremely difficult to attain.

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[1)]The expression (34) contains no such divergence.
[2)]We recall that the single-particle spectrum may not exist in the singular region (cf. the Conclusion).
[3)]A change of sign of μ corresponds only to a change of basis and, therefore, has no physical significance.
[4)]Our calculation had the purpose of matching with the singular region, and we have therefore considered the isotropic model with $f = 0$. In calculations for the second region it is necessary to use the true "free" Green function.

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Nonlinear size effect in bismuth

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It is shown that when a semimetal is placed in a constant magnetic field parallel to its surface and is exposed to radio waves, a system of dc spikes is produced in addition to the spikes of the high-frequency current in the interior of the sample. The appearance of the dc spikes is due to the change in the length of the electron path in the skin layer under the influence of the magnetic field of the wave itself. In samples of limited dimensions, the presence of a system of dc spikes, the distance between which depends on the magnitude of the constant magnetic field, should lead to the appearance of a nonlinear size effect. This effect is experimentally investigated in bismuth.

Several authors[1-3] have shown recently that, at least in semimetals, various nonlinear phenomena can be observed even at relatively small electromagnetic-wave powers. In particular, direct current is produced in a skin layer under the influence of electromagnetic radiation.

It is known that spikes of high-frequency current can be produced in the interior of a metal[4] in the presence of a static magnetic field under the conditions of the anomalous skin effect, and are due to the individual motion of the electrons of the extremal sections of the Fermi surface. One can expect the presence of direct current concentrated at the surface of the sample to lead analogously to the onset of direct-current spikes in the interior of the sample.

Let us examine the qualitative aspect of the phenomenon. We confine ourselves to the case of low frequencies $\omega\tau \ll 1$, $\delta/r \ll 1$ (ω is the frequency of the electromagnetic wave, τ is the electron relaxation time, δ is the depth of the skin layer, and r is the Larmor radius). In this case we can assume that the electron move in a quasistatic but spatially strongly inhomogeneous electromagnetic field. Since the phase shift between the alternating electric field E and the magnetic field H_1 differs from $\pi/2$, superposition of the magnetic field of the wave itself on the static magnetic field H_0 causes the two half-cycles of the alternating current to become unequal, owing to the change in the path length of the effective electrons in the skin layer. As a result, an electric current proportional to H_1E and having a dc component appears in the skin layer.

It was easy to estimate the density j_2 of the direct current if the field H_0 is parallel to the surface of the metal. The high-frequency current j_1 is proportional to $\sigma_0 E\delta r^{-1}$, where σ_0 is the static conductivity. The influence of the magnetic field is estimated from the relation $j_2 \sim H_1 dj_1/dH_0$. Using the estimate $H_1 \sim cE/\omega\delta$, we obtain $j_0 \sim \sigma_0 E(\omega\tau)^{-1} eEl\epsilon_F^{-1}$ where l is the electron mean free path and ϵ_F is the Fermi energy.

In a magnetic field parallel to the surface of the metal, the electrons of the extremal section of the Fermi surface, after passing through the skin layer, are again focused at a depth 2r as they move along the trajectories and produce at this depth a direct-current spike. At the same distance from the surface there is a spike of fields H_1 and E, which leads to the appearance of direct current at the depths 2r and 4r. Thus, a system of direct-current spikes is produced. It is clear from the foregoing arguments that a similar system of spikes is produced also for the current j_2 at the frequency 2ω.

In a sample of limited dimensions, the presence of a system of dc spikes, the distance between which is determined by the value of the external magnetic field H_0, should lead to the appearance of a nonlinear size effect, similar to the radio-frequency size effect[4]. The present article is devoted to this phenomenon.

1. THEORY

We shall show that a more rigorous allowance for the change in the electron path length in the skin layer to the proper magnetic field of the wave leads to the appearance in the skin layer of a direct current j_2 and to the appearance in the interior of the sample of dc spikes. Assume that an electromagnetic wave is incident on the surface of a metal occupying the half-space $z > 0$. The electric field $\mathbf{E}$ of the wave is parallel to the x axis. A static magnetic field $\mathbf{H_0}$ is parallel to the y axis. We assume that the quasistationarity condition $\omega\tau \gg 1$ and the strong-magnetic-field condition $\Omega\tau \gg 1$ are satisfied ($\Omega = eH_0/mc$ is the cyclotron frequency). As the model of the Fermi surface of the metal we use a cylindrical surface $p_x^2 + p_z^2 = p^2$ (a similar model can be used in the case of bismuth, the electronic part of the Fermi surface of which consists of three strongly elongated ellipsoids). The length of the cylinder is assumed to be p_H, so that the electron density is $n = 2\pi p_H p^2 h^{-3}$.

To find the response of the electron system to an external perturbation, it is necessary to solve the Boltzmann kinetic equation. In the τ-approximation we have

$$v_z\frac{\partial f}{\partial z} + \Omega\frac{\partial f}{\partial\varphi} + \tau^{-1}f = g, \qquad (1)$$

where $df_0/d\epsilon$ is the nonequilibrium part of the distribution function, $\varphi = \Omega t$ is the dimensionless time of motion along the orbit, and g describes the external perturbation.

When solving the kinetic equation we shall use the methods developed in the theory of anomalous penetration of the electromagnetic field into a metal[4]. Neglecting the field E_z and the boundary conditions for the distribution function at $z = 0$, and continuing the field E in even fashion to the region $z < 0$, we employ the Fourier transformation:

$$A_k = \int_{-\infty}^{\infty} dz A(z) e^{-ikz}, \quad A(z) = (2\pi)^{-1}\int_{-\infty}^{\infty} dk A_k e^{ikz}. \qquad (2)$$

We can then easily obtain the electric current j_k