(released late due to instructor illness)

Quasiparticle transport in a superconductor

1. Electron tunneling.

Consider two metals that can be in a normal or in a superconducting state coupled through a tunnel junction,

$$\mathcal{H}_t = \sum_{\sigma,k,q} T_{k,q} c^{+}_{k,\sigma} c_{q,\sigma} + \text{h.c.} \tag{1}$$

where $c_{q,\sigma}$ and $c_{k,\sigma}$ are Fermi operators of an electron in the metal and in the superconductor, respectively.

a) Consider tunneling current in the presence of voltage V applied across the barrier. Using the Golden Rule $dW = \frac{2\pi}{\hbar}|\langle f|\mathcal{H}_t|i\rangle|^2$, evaluate the rate of transitions from material 1 to the material 2 and show that

$$I_{1\to 2} = A\int_{-\infty}^{\infty} |T|^2 N_1(E) f(E) N_2(E+eV)\left[1 - f(E+eV)\right] dE \tag{2}$$

with $N_{1,2}(E)$ the density of states $d\mathcal{N}/d\epsilon$ in both materials, $f(E)$ the Fermi distribution. Here A is a proportionality constant and $|T|^2 = \sum_{\epsilon_k,\epsilon_q} |T_{k,q}|^2$.

b) Following the route that has led to Eq.(2), find a similar expression for the current $I_{2\to 1}$ from material 2 into the material 1. For the total tunneling current $I = I_{1\to 2} - I_{2\to 1}$ obtain

$$I = A\int_{-\infty}^{\infty} |T|^2 N_1(E) N_2(E+eV)\left[f(E) - f(E+eV)\right] dE \tag{3}$$

c) Verify that for a pair of normal metals, with constant density of states each, the tunneling current (3) obeys Ohm's law, $I = GV$.

d) Consider tunneling between a normal metal and a superconductor. Analyze the expression for the current and plot I vs. V at low temperature and at $T = 0$. Show that the so-called *tunneling density of states* $W(V) = dI/dV$ at zero temperature is proportional to the BCS quasiparticle density of states

$$W(V) \propto N(E) = \nu_0 \Delta / \left(E^2 - \Delta^2\right)^{1/2}_{E=eV} \tag{4}$$

2. Andreev reflection.

Charge transport through a clean normal metal-superconductor interface can be described by Bogoliubov-deGennes equation with position dependent pairing amplitude $\Delta(r)$,

$$E\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \mathcal{H} & \Delta(r) \\ \Delta^*(r) & -\mathcal{H}^* \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{5}$$

where $\mathcal{H}$ is a single-particle Hamiltonian of noninteracting fermions.

a) Consider a one-dimensional problem with a step-like pairing function $\Delta(x<0) = 0$, $\Delta(x>0) = \Delta$, and $\mathcal{H} = -\frac{\hbar^2}{2m}\partial_x^2 - E_F$. Consider scattering state of an electron incident on the NS interface with the energy below the BCS gap, $|E - E_F| < \Delta$. Show that in

the superconductor the solution is an evanescent wave, and in the metal it describes a reflected hole.
b) Now consider incident electron with the energy slightly above the gap, $|E - E_F| > \Delta$. Describe the result of scattering of such an electron.
c) Generalize the result of part a) to a 3D system. Consider planar normal metal-superconductor interface with an electron incident at an angle θ to normal. Find the direction of the outgoing hole.

Fermi liquid theory

3. Thermodynamic functions, specific heat.

a) Thermodynamic potential of the ideal Fermi gas can be evaluated as

$$\Omega = -T \int \ln\left(1 + e^{-\beta\epsilon_p}\right) d^3p/(2\pi\hbar)^3 \tag{6}$$

Starting from this expression, show that specific heat is a linear function of temperature at $T \ll E_F$. Find the proportionality constant in the relation $C = \gamma T$.

b) Consider the thermodynamic potential using the particle-hole oscillator representation. We are going to check if it gives the same result as the canonical Fermi representation. In the absence of interactions,

$$\mathcal{H} = \sum_{\mathbf{k},\mathbf{p}\in R_{\mathbf{k}}} \left(\frac{1}{2}\pi^*_{\mathbf{k},\mathbf{p}}\pi_{\mathbf{k},\mathbf{p}} + \frac{1}{2}\omega^2_{\mathbf{p},\mathbf{k}}\phi^*_{\mathbf{k},\mathbf{p}}\phi_{\mathbf{k},\mathbf{p}}\right) \tag{7}$$

with $\omega_{\mathbf{p},\mathbf{k}} = (\mathbf{p}+\mathbf{k})^2/2m - \mathbf{p}^2/2m$. Apply the formula for the thermodynamic potential of an ensemble of free bosons,

$$\Omega = T\sum_\alpha \ln\left(1 - e^{-\beta\hbar\omega_\alpha}\right) = T \sum_{\mathbf{k},\mathbf{p}\in R_k} \ln\left(1 - e^{-\beta\hbar\omega_{\mathbf{p},\mathbf{k}}}\right)$$

with $\omega_{\mathbf{p},\mathbf{k}} = \mathbf{v}\cdot\mathbf{k} + \mathbf{k}^2/2m$, $\mathbf{v} = \mathbf{p}/m$, and the crescent domain R_k defined in lecture as an overlap of a displaced Fermi sphere complement $|\mathbf{p}+\mathbf{k}| > p_0$ with the undisplaced Fermi sphere $|\mathbf{p}| < p_0$. Compare with the result of part a). To simplify analysis, consider only low temperatures, and find the specific heat for $T \ll E_F$.

c) Consider thermodynamic functions of the system of interacting fermions using the oscillator representation,

$$\mathcal{H}_{int} = \sum_{\mathbf{k},\mathbf{p},\mathbf{p}'\in R_{\mathbf{k}}} V_{\mathbf{k}}\omega^{1/2}_{\mathbf{p},\mathbf{k}}\omega^{1/2}_{\mathbf{p}',\mathbf{k}}\phi^*_{\mathbf{k},\mathbf{p}}\phi_{\mathbf{k},\mathbf{p}'} \tag{8}$$

Compare with the noninteracting case. Like in part b), do the calculations assuming $T \ll E_F$.

4. Screening

Consider screening of an external potential in a 3D electron gas with Coulomb repulsion $V_{\mathbf{k}} = \int (e^2/|\mathbf{r}|)e^{i\mathbf{kr}}d^3r = 4\pi e^2/\mathbf{k}^2$ between electrons. Show that for a slowly varying potential, the screened potential is described by

$$V_{\mathbf{k}} = \frac{\mathbf{k}^2}{\mathbf{k}^2 + r_s^{-2}} V^{ext}_{\mathbf{k}}, \quad r_s^{-2} = 4\pi\nu e^2 \tag{9}$$

with ν the density of states at the Fermi level. The quantity r_s is the screening radius, as can be seen from the form of the screened Coulomb potential $\frac{1}{r}e^{-r/r_s}$.