Interacting Fermions

1. Short-range interaction.

a) Consider Schrödinger equation in $D = 1$ for two spinless fermions moving in an external potential $U(x) = \frac{1}{2}m\omega^2x^2$ and interacting via a short-range potential $\lambda\delta(x - x')$. Find the energy spectrum and the eigenstates for this two-body problem.
b) Generalize the result of part a) to any number of fermions.
c) Generalize the result of part a) to an arbitrary external potential $U(x)$ and arbitrary space dimension.
d) Prove that the energies and states of spinless fermions with short-range interaction are the same as without interaction by starting from the second-quantized hamiltonian

$$\mathcal{H} = \int \left(\hat{\psi}^+(x) \left(-\frac{\hbar^2}{2m} + U(x) \right) \hat{\psi}(x) + \frac{\lambda}{2} \hat{\psi}^+(x)\hat{\psi}^+(x)\hat{\psi}(x)\hat{\psi}(x) \right) dx \tag{1}$$

and using the algebra of the field operators $\hat{\psi}(x)$ and $\hat{\psi}^+(x)$.

2. Cooper pairs.

To understand the origin of pairing in the presence of attractive interaction in a Fermi system, Cooper proposed to replace the many-body problem by a toy model of two particles at the Fermi level. To take into account that the states below the Fermi level are filled, and thus are 'dynamically inaccessible,' the Hilbert space of states for each of the two particles in this model consists of all single-particle states with energies above the Fermi level. The Hilbert space for the two particles consists of the states

$$|\mathbf{p}, \alpha, \mathbf{p}', \beta\rangle = a^+_{\mathbf{p},\alpha} a^+_{\mathbf{p}',\beta}|0\rangle\,, \quad |\mathbf{p}|, |\mathbf{p}'| > p_F\,, \quad \alpha, \beta = \uparrow, \downarrow \tag{2}$$

The energy of the state (2), in the absence of interactions, is $E_{\mathbf{p}} + E_{\mathbf{p}'}$, where $E_{\mathbf{p}} = \mathbf{p}^2/2m - p_F^2/2m$.
a) Consider short-range interaction

$$\mathcal{H}_{int} = \frac{1}{2}\lambda \int \sum_{\alpha,\beta=\uparrow,\downarrow} \hat{\psi}^+_\alpha(x)\hat{\psi}^+_\beta(x)\hat{\psi}_\beta(x)\hat{\psi}_\alpha(x)dx \tag{3}$$

Find the matrix elements of the interaction (3) between the states (2)
b) Show that the total momentum $\mathbf{P} = \mathbf{p} + \mathbf{p}'$ is conserved, and thus the two-particle Hilbert space decouples into subspaces with fixed value of $\mathbf{P}$.
c) Let us fix $\mathbf{P} = 0$, i.e. consider the subspace of two-particle states of the form

$$|\mathbf{p}, \alpha, -\mathbf{p}, \beta\rangle = a^+_{\mathbf{p},\alpha} a^+_{-\mathbf{p},\beta}|0\rangle\,, \quad |\mathbf{p}| > p_F\,, \quad \alpha, \beta = \uparrow, \downarrow \tag{4}$$

Show that the interaction (3) projected on (or, acting within) this subspace is an operator of rank one. In other words, up to a constant, it maps all vectors onto one specific vector.
d) Suppose the hamiltonian has the form $\mathcal{H} = A + B$, where A is a diagonal matrix with the spectrum E_i, and B is an operator of rank one with vector $|u\rangle = (u_1, u_2, u_3, ...)$

corresponding to its only nonzero eigenvalue λ. Show that the operator $(\epsilon - A)^{-1} B$ is of rank one for $\epsilon \neq E_i$. Use this fact to obtain an algebraic equation for the spectrum of $\mathcal{H}$.
e) Apply the result of part d) to the problem of Cooper pair with an attractive interaction $\lambda < 0$ and zero total momentum $\mathbf{P}$. Show that the spectrum of the problem consists of a continuum of states with positive energies, and of one discrete state with negative energy. (To cut a divergence in the eigenvalue equation, you may assume that the interaction strength is gradually decreasing at large energies, $\lambda \to \lambda \exp(-E_{\mathbf{p}}/E_*)$, with $E_* \sim E_F$.