Bose condensation

1. Vortices.

a) Starting from the superflow equations away from singularities, $\mathbf{v} = \frac{\hbar}{m}\nabla\theta$, $\oint \mathbf{v}\cdot d\mathbf{r} = 2\pi\frac{\hbar}{m}l$ with integer l, show that the velocity field $\mathbf{v}(\mathbf{r})$ can be found from a 'magnetostatics problem'

$$\nabla\cdot\mathbf{v} = 0\,,\quad \nabla\times\mathbf{v} = 2\pi\frac{\hbar}{m}\mathbf{j}(\mathbf{r}) \tag{1}$$

Here $\mathbf{j}(\mathbf{r})$ is an auxiliary line current that flows along vortex cores, and for each vortex takes an integer value equal to the quantized circulation l.

Argue that the velocity field of several vortices can be obtained from superposition principle. Consider two vortices of unit circulation, aligned parallel to each other and separated by a distance d. Find the velocity $\mathbf{v}(\mathbf{r})$, the phase $\theta(\mathbf{r})$, and the interaction energy of the two vortices. (Consider two situations, with vortices of the same sign and of opposite signs.)

b) Consider a vortex near the wall of a container with superfluid. The vortex is aligned parallel to the wall at a distance d from it. The flow around the vortex is distorted due to the presence of the wall. Show that this distortion can be characterized by introducing an image vortex on the other side of the wall, along with extending the flow to the entire space (i.e. removing the wall). What is the sign of the image vortex? Find the superflow velocity $\mathbf{v}(\mathbf{r})$ and the interaction energy of the vortex and the wall. Is this interaction attractive or repulsive?

c) For a generic vortex configuration, starting from Eq. (1), derive 'Biot-Savart formula'

$$\mathbf{v}(\mathbf{r}) = \sum_\alpha \frac{\hbar l_\alpha}{2m}\int \frac{d\mathbf{a}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{2}$$

for superflow velocity. Here the integral is taken over vortex core lines, the sum is taken over all vortices of circulation l_α each.

Consider a circular vortex ring of radius R. Each portion of the ring is sitting in a flow induced by other parts of the ring. As a result, the ring propels itself as a whole, and moves without changing the shape and radius. Find the velocity of self-propelled ring motion. Describe the dependence of ring velocity on the radius. Show that smaller rings move faster than larger rings.

d) Consider a vortex in an infinite system. Suppose that the vortex core line is slightly displaced relative to its initial completely straight configuration. Analyze how this displacement evolves in time and propagates along the core line. For a small displacement amplitude, linearize the problem and find the dispersion relation for long-wavelength vortex oscillations.

2. Collective modes of trapped BEC.

a) Consider Bose gas in a harmonic trap,

$$\mathcal{H} = \int \left[\hat\phi^+(r)\left(-\frac{\hbar^2}{2m}\nabla_r^2 + U(r)\right)\hat\phi(r) + \frac{\lambda}{2}\hat\phi^+(r)\hat\phi^+(r)\hat\phi(r)\hat\phi(r)\right] d^3r \tag{3}$$

with the trap potential $U(r) = \frac{1}{2}m\omega_0^2\mathbf{r}^2 = \frac{1}{2}m\omega_0^2(x^2+y^2+z^2)$. Collective modes of this system, in general, depend on the temperature of the gas and on the interaction strength. However, for one special mode, sometimes called Kohn mode, that corresponds to the center of mass motion of the gas, the behavior is universal. Show that the center of mass operator

$$\hat{R} = \int \hat{\phi}^{+}(r)\mathbf{r}\phi(x)d^3r \tag{4}$$

obeys $\ddot{\mathbf{R}} = -\omega_0^2\mathbf{R}$, and thus the dynamics of $\mathbf{R}$ is characterized by frequency ω_0 irrespective of the quantum state of the gas.

b) Collective modes of Bose condensate can be studied by using Gross-Pitaevskii equation, as the long-wavelength dynamics of the density and phase variables,

$$(i): \quad \partial_t n = -\nabla(n\mathbf{v}), \quad \mathbf{v} = \frac{\hbar}{m}\nabla\theta; \qquad (ii): \quad \partial_t\theta = -\mu/\hbar, \quad \mu = U(r) + \lambda n + \frac{1}{2}m\mathbf{v}^2 \tag{5}$$

Linearize these equations for small fluctuations in a steady state characterized by density $n(r)$, taking the limit of long wavelength, and obtain the wave equation for collective modes,

$$\ddot{\theta} = \frac{\lambda}{m}\nabla(n\nabla\theta) \tag{6}$$

Check that for a system at uniform density the collective modes have the same sound-like dispersion as Bogoliubov quasiparticles at low energy.

c) Consider collective modes in a trapped BEC sample of radius R at $T = 0$, with density distribution $n(r) = \frac{1}{\lambda}(\mu - U(r)) = \frac{m\omega_0^2}{2\lambda}(R^2 - r^2)$,. In this case, since n varies in space, the wave equation (6) cannot be solved by Fourier transform. (No plane waves in a finite system!) Instead one has to look for normal modes of the problem (6) which we rewrite as

$$-\omega^2\theta = \frac{\omega_0^2}{2}\left(\left(R^2 - r^2\right)\nabla^2\theta - 2\mathbf{r}\cdot\nabla\theta\right) \tag{7}$$

Show that there is a class of special solutions with special dependence $\theta(r) = r^{l+1}Y_{l,m}(\alpha,\beta)$, where $Y_{l,m}$ are spherical harmonics of the spherical angles α, β. (Note that the functions of this form satisfy Laplace equation $\nabla^2 f = 0$.) Find the frequency of oscillations as a function of the spherical harmonic number l. Which of these modes correspond to the center of mass motion discussed in part a)?

d) Find the spectrum of all collective modes of the problem (7). (Hint: Use the variable $f(\mathbf{r}) = r\theta(\mathbf{r})$. For a particular spherical harmonic, write $f(\mathbf{r}) = f(r)Y_{l,m}(\alpha,\beta)$, then look for a solution in the form of power series

$$f(r) = \sum_{j=l+1} a_j r^j \tag{8}$$

and use Eq.(7) to obtain a recursion relation for the coefficients a_j.)