## Bose condensation

**1. Quasiparticles.**

Consider a Bose gas at $T = 0$ with one quasiparticle with momentum $\mathbf{p} \neq 0$ added on the top. Quasiparticle state can be obtained by applying the quasiparticle creation operator to the nonideal Bose gas ground state:

$$|1_{\mathbf{p}}\rangle = \widehat{b}^{+}_{\mathbf{p}}|0\rangle \tag{1}$$

where $\widehat{b}^{+}_{\mathbf{p}} = \cosh\theta_{\mathbf{p}}\widehat{a}^{+}_{\mathbf{p}} - \sinh\theta_{\mathbf{p}}\widehat{a}_{\mathbf{p}}$.

How many particles are contained in one quasiparticle? To find out, take the number operator $\widehat{N} = \sum_{\mathbf{k}} \widehat{a}^{+}_{\mathbf{k}}\widehat{a}_{\mathbf{k}}$ of the original particles and evaluate the difference

$$\bar{N}_{\mathbf{p}} = \langle 1_{\mathbf{p}}|\widehat{N}|1_{\mathbf{p}}\rangle - \langle 0|\widehat{N}|0\rangle \tag{2}$$

(Be careful: $\widehat{a}_{\mathbf{p}}|0\rangle \neq 0$, instead $\widehat{b}_{\mathbf{p}}|0\rangle = 0$.) Express the answer in terms of the Bogoliubov angle $\theta_{\mathbf{p}}$. Compare the situation at high and low quasiparticle energy and interpret the result.

**2. Landau criterion for superfluidity.**

A superflow state of a Bose condensate having velocity $\mathbf{v}$ is characterized by macroscopic occupancy of a state with nonzero momentum $\mathbf{p} = m\mathbf{v}$. The many body state can be constructed by generalizing the scheme used to describe stationary condensate:

$$|\Phi_{\mathbf{v}}\rangle = \exp\left(\sqrt{V}(\phi(x)\widehat{a}_{\mathbf{p}} - \bar{\phi}(x)\widehat{a}^{+}_{\mathbf{p}})\right), \quad \phi(x) = \phi\,\exp\left(\frac{i}{\hbar}\mathbf{p}\mathbf{x}\right) \tag{3}$$

a) Starting from this state, consider the expectation value $\langle\Phi_{\mathbf{v}}|\mathcal{H} - \mu\widehat{N}|\Phi_{\mathbf{v}}\rangle$ and, by minimizing energy in $\phi$, obtain the chemical potential $\mu$ of the superflow state. How does $\mu$ depend on the superflow velocity $\mathbf{v}$?

b) Consider elementary excitations (quasiparticles) in the superflow state. The Bose gas hamiltonian expanded up to second order in $a_{\mathbf{k}}$, $a^{+}_{\mathbf{k}}$, has the form

$$\mathcal{H} = E_0 + \sum_{\mathbf{k}\neq 0}\left(\epsilon^{(0)}_{\mathbf{k}} - \mu + 2\lambda|\phi|^2\right)a^{+}_{\mathbf{k}}a_{\mathbf{k}} + \frac{1}{2}\lambda\sum_{\mathbf{k}\neq 0}\left(\phi^2 a^{+}_{\mathbf{k}}a^{+}_{2\mathbf{p}-\mathbf{k}} + \bar{\phi}^2 a_{\mathbf{k}}a_{2\mathbf{p}-\mathbf{k}}\right) \tag{4}$$

To diagonalize this hamiltonian, group together the states with momenta $\mathbf{k}$ and $2\mathbf{p} - \mathbf{k}$,

$$\widehat{b}_{\mathbf{k}} = \cosh\theta_{\mathbf{k}}\widehat{a}_{\mathbf{k}} - \sinh\theta_{\mathbf{k}}\widehat{a}^{+}_{2\mathbf{p}-\mathbf{k}}, \quad \widehat{b}^{+}_{\mathbf{k}} = \cosh\theta_{\mathbf{k}}\widehat{a}^{+}_{\mathbf{k}} - \sinh\theta_{\mathbf{k}}\widehat{a}_{2\mathbf{p}-\mathbf{k}} \tag{5}$$

Find the parameters $\theta_{\mathbf{k}}$ that diagonalize the hamiltonian, and obtain the quasiparticle dispersion relation $E(\mathbf{k})$. (Hint: Don't let yourself be dragged into long calculation, — the result can be more or less read off the solution for stationary BEC with slight adjustments.)

Find the critical superflow velocity $v_c$ above which the energy of quasiparticles $E(\mathbf{k})$ can become negative. Landau argued that the superfluid can sustain nondissipative flows with velocities $v < v_c$, and in this way he could explain the phenomenon of critical velocity observed in superfluid $^4He$. At $E(\mathbf{k}) > 0$ the quasiparticles cannot be created spontaneously, while at $v > v_c$ the flow is accompanied by massive quasiparticle creation, and is thus dissipative. Find the critical velocity $v_c$ for nonideal Bose gas.

c) Can you interprete the result of part b) for quasiparticle dispersion in superflow from the point of view of a Galilean transformation? Note that the microscopic hamiltonian is invariant with respect to changing the reference frame from stationary to moving, $\mathbf{x}' = \mathbf{x} + \mathbf{v}t$, $t' = t$. Show that for an excitation with frequency $\omega$ and wavevector $\mathbf{k}$ this yields $\omega' = \omega - \mathbf{k}\mathbf{v}$, $\mathbf{k}' = \mathbf{k}$. How is the quasiparticle energy changed under a Galilean transformation?

**3. Condensate depletion.**

a) In a nonideal Bose gas at $T = 0$ only a fraction of all the particles is found in the condensate. The reduction of condensate density due to interactions is called "condensate depletion." (An extreme example is prodided by $^4$He, where the majority of the particles — more than 90% — are not in the condensate. To estimate this effect in a weakly nonideal Bose gas, find the expectation value of the total density

$$\hat{n} = \hat{n}_0 + V^{-1} \sum_{\mathbf{k}\neq 0} \hat{n}_{\mathbf{k}}\,, \quad \hat{n}_{\mathbf{k}} = \hat{a}^+_{\mathbf{k}} \hat{a}_{\mathbf{k}} \tag{6}$$

over the ground state. The first term gives the condensate density $n_0 = \langle a_0^+ \hat{a}_0 \rangle$, while the second term gives the density of the out-of-condensate particles. Find the depletion factor $(n - n_0)/n$ dependence on the coupling constant $\lambda$.

b) Consider the correlator of the field operators $R(x, x') = \langle 0|\hat{\phi}^+(\mathbf{x})\hat{\phi}(\mathbf{x}')|0\rangle$. Show that it is related to the particle number distribution as $R(\mathbf{x} - \mathbf{x}') = \sum_{\mathbf{k}} \langle 0|n_{\mathbf{k}}|0\rangle e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')}$. Describe the behavior of $R(\mathbf{x} - \mathbf{x}')$ as a function of point separation $\mathbf{x} - \mathbf{x}'$. Find the limits at $|\mathbf{x} - \mathbf{x}'| \to \infty$ and at $\mathbf{x} = \mathbf{x}'$. Estimate the length scale $\xi$, called *BEC healing length*, at which the crossover from $R(0)$ to $R(\infty)$ takes place.

**4. Thermodynamics of a Bose gas.**

Thermodynamic quantities of Bose-condensed gas can be found by treating the system as a gas of noninteracting Bogoliubov quasiparticles obeying Bose statistics. The thermodynamic potential of the system is

$$\Omega \equiv -T \ln Z = T \int \ln\left(1 - e^{-\beta E(\mathbf{k})}\right) \frac{d^3k}{(2\pi)^3}\,, \quad E(\mathbf{k}) = \sqrt{\epsilon^{(0)}(\mathbf{k})\left(\epsilon^{(0)}(\mathbf{k}) + 2\lambda n\right)} \tag{7}$$

with $\epsilon^{(0)}(\mathbf{k}) = \hbar^2\mathbf{k}^2/2m$.

a) Show that simple analytical results for the thermodynamic potential $\Omega$ can be obtained at very low temperatures, $T \ll T_\lambda \equiv \lambda n$ and at moderately high temperatures, $T_\lambda \ll T \le T_{BEC}$. (Hint: Given the temperature, low or high, simplify the form of $E(\mathbf{k})$ by ignoring $\epsilon^{(0)}(\mathbf{k})$ compaired to $\lambda n$, or vice versa.)

b) Find the entropy, the specific heat, and the normal component density $n(T)$ in the above two temperature intervals. Compare with the ideal Bose gas.