Squeezed states

1. Squeeze operators.
Consider a unitary operator $U(\theta) = \exp\left(\theta\left(\hat{a}\hat{a} - \hat{a}^+\hat{a}^+\right)/2\right)$.
a) Prove that

$$U^+(\theta)\hat{a}U(\theta) = \cosh\theta\hat{a} - \sinh\theta\hat{a}^+ , \quad U^+(\theta)a^+U(\theta) = \cosh\theta\hat{a}^+ - \sinh\theta\hat{a} \tag{1}$$

(Hint: use the operator expansion theorem, Problem 1 a), PS#1). From that derive the transformation rule for the coordinate and momentum operators,

$$U^+(\theta)\hat{q}U(\theta) = e^{-\theta}\hat{q}\,, \quad U^+(\theta)\hat{p}U(\theta) = e^{\theta}\hat{p} \tag{2}$$

b) To show that the operator $U(\theta)$, applied to the vacuum state $|0\rangle$, generates a squeezed state, calculate the coordinate and momentum uncertainty, $\langle\delta q^2\rangle$, $\langle\delta p^2\rangle$, and show that the uncertainty product equals $\frac{1}{2}\hbar$, independent of θ.
c) To characterize the time evolution of the state $\psi_0 = U(\theta)|0\rangle$, formally given by $\psi(t) = e^{-i\mathcal{H}t/\hbar}\psi_0$, find the variance matrix

$$\begin{pmatrix} \langle\hat{q}^2\rangle_{\psi(t)} & \langle\frac{1}{2}\{\hat{q},\hat{p}\}_+\rangle_{\psi(t)} \\ \langle\frac{1}{2}\{\hat{q},\hat{p}\}_+\rangle_{\psi(t)} & \langle\hat{p}^2\rangle_{\psi(t)} \end{pmatrix} \tag{3}$$

time dependence. (Here the expectation values $\langle...\rangle_{\psi(t)} = \langle\psi(t)|...|\psi(t)\rangle$, and $\{\hat{q},\hat{p}\}_+ = \hat{q}\hat{p} + \hat{p}\hat{q}$.)

2. Time-dependent states of a harmonic oscillator.
Consider a harmonic oscillator with a time-dependent frequency,

$$\mathcal{H}(t) = \frac{p^2}{2m} + \frac{m\omega^2(t)}{2}q^2 \tag{4}$$

a) Suppose that $\omega(t)$ is a given function of time. Look for a solution of the Schrödinger evolution equation $i\hbar\partial_t\psi = \mathcal{H}(t)\psi$ of a gaussian form,

$$\psi(q,t) = A(t)\exp(-\alpha(t)q^2/2) \tag{5}$$

From the consistency requirement for such an ansatz, obtain a nonlinear differential equation that relates the time-dependent $\alpha(t)$ with $\omega(t)$.
b) Show that a squeezed state time evolution can be obtained from the condition

$$\left(P(t)\hat{q} - Q(t)\hat{p}\right)\psi(t) = 0 \tag{6}$$

where $P(t)$ and $Q(t)$ are complex solutions of the classical Hamilton equations $\dot{Q} = P/m$, $\dot{P} = -m\omega^2 Q$.

The equations for P and Q are linear, while the equation for $\alpha(t)$ found in part a) is nonlinear. To establish a connection between the two methods, find a substitution that turns the equation for $\alpha(t)$ into a linear equation.

c) Consider a harmonic oscillator initially in the ground state. The parabolic potential is abruptly removed at $t = 0$, and then restored at $t = \tau$. Find the state at $0 < t < \tau$ and at $t > \tau$.

d) A popular practical method of producing squeezed states involves parametric resonance which takes place when the parameters of the oscillator are externally varied at a frequency close to twice the unperturbed normal frequency,

$$\omega^2(t) = \omega_0^2 + \lambda \cos \Omega t \,, \quad \Omega = 2\omega_0 \tag{7}$$

Taking the oscillator initially in the ground state and assuming small λ, obtain the time dependence of $P(t)$, $Q(t)$.

A note on weakly perturbed oscillator: At small λ, it is convenient to look for a solution of the equation $\ddot{Q} + \omega^2(t)Q = 0$ in the form $Q(t) = A(t)\cos\omega_0 t + B(t)\sin\omega_0 t$. For unperturbed harmonic oscillator, at $\lambda = 0$, the solution is given by constant A, B. Accordingly, for a weakly perturbed oscillator, the leading time-dependence $A(t)$, $B(t)$ should be slow. Based on this intuition, derive the differential equations for $A(t)$, $B(t)$ by discarding the rapidly oscillating terms (argue that their effect is negligible).

To analyze wavepacket evolution, from the solution $P(t)$, $Q(t)$ find the parameter $\alpha(t)$. Qualitatively, sketch the width of the wavepacket as a function of time.

3. The phase-space density of a squeezed state.

a) Show that the Wigner function $W(q,p)$ of a squeezed state is a gaussian distribution in the phase space.

b) For a general gaussian distribution $P(x) \propto \exp\left(-\frac{1}{2}\sum_{ij=1}^{n} D_{ij}x_i x_j\right)$ of an n-component variable x_i, show that

$$D_{ij} = \left(M^{-1}\right)_{ij} \,, \quad M_{ij} = \langle x_i x_j \rangle \tag{8}$$

In other words, the matrix D is fully characterized by the variance matrix M.

Consider the Wigner function $W(q,p)$ of a squeezed state. Using the result of Problem 1, part c), find the time dependence $M(t)$ and $D(t)$.

c) For the states obtained in Problem 2, parts c) and d), reconstruct and qualitatively describe the time evolution of the phase-space distribution $W(q,p)$. You may find it useful to use numerics for visualization.