Field integral. Bosons and fermions.

Reading: M. Stone, The Physics of Quantum Fields

1. Bose gas

Consider a saddle point approximation for the functional integral theory of a weakly nonideal Bose gas with a repulsive interaction,

$$Z = \int d[\bar{\phi}, \phi] e^{\frac{i}{\hbar} S}, \quad S = \iint \left(\frac{i}{2} \bar{\phi} \partial_t \phi + c.c. - \mathcal{H}(\bar{\phi}, \phi) \right) d^3x dt\,, \tag{1}$$

$$\mathcal{H}(\bar{\phi}, \phi) = \bar{\phi} \left(-\frac{1}{2} \nabla^2 - \mu \right) \phi + \frac{1}{2} \lambda (\bar{\phi}\phi)^2\,, \quad \lambda > 0$$

a) Show that the saddle point equation $\delta S / \delta \bar{\phi} = 0$ gives the Gross-Pitaevsky (GP) equation.

b) Perform a transformation to imaginary time, $t \to -it$, and consider fluctuations about the stationary solution of the GP equation, $\phi = \sqrt{n} e^{i\theta}$, Evaluate the fluctuation determinant contribution to Z at finite temperature, and show that the answer agrees with what is expected from Bogoliubov quasiparticles, treated as free bosons.

2. Ideal Fermi gas.

Consider functional integral representation of the ideal spinless Fermi gas,

$$Z = \int d[\bar{\psi}, \psi] \exp \left(i \iint \left(\frac{i}{2} \bar{\psi} \partial_t \psi + c.c. - \bar{\psi} \widehat{H}_0 \psi \right) d^3x dt \right)\,, \quad \widehat{H}_0 = -\frac{1}{2} \nabla^2 - \mu \tag{2}$$

To find the partition function at finite temperature, go to imaginary time, $t \to -it$, and integrate over the anticommuting variables ψ and $\bar{\psi}$,

$$Z = \det \left(\partial_t - \widehat{H}_0 \right) \tag{3}$$

Evaluate the determinant, taking into account that fermionic fields obey an antiperiodic boundary condition, $\psi(t + \beta) = -\psi(t)$. (In Fourier components, that is a sum over Matsubara frequencies, $\psi(t) = \sum_n e^{i\omega_n t} \psi_n$, $\omega_n = (2n+1)\pi T$.) Show that the result agrees with the partition function of an ideal Fermi gas.

3. BCS pairing. Functional integral approach.

Consider a Fermi gas with an attractive interaction,

$$Z = \int d[\bar{\psi}, \psi] \exp \left(i \iint \sum_{\alpha = \uparrow, \downarrow} \left(\frac{i}{2} \bar{\psi} \partial_t \psi + c.c. - \bar{\psi} \widehat{H}_0 \psi + \lambda \bar{\psi}_\downarrow \bar{\psi}_\uparrow \psi_\uparrow \psi_\downarrow \right) d^3x dt \right)\,, \tag{4}$$

$$\widehat{H}_0 = -\frac{1}{2} \nabla^2 - \mu\,, \quad \lambda < 0$$

This problem describes BCS pairing. It is very convenient to construct the mean field theory using the functional integral representation.

a) Perform a transformation to imaginary time, $t \to -it$, and apply the Habbard-Stratonovitch identity: decouple the quartic term with the help of integration over a complex-valued field $\Delta(\mathbf{x}, t)$:

$$Z = \int d[\bar{\Delta}, \Delta] e^{-\frac{1}{2\lambda} \iint |\Delta(\mathbf{x}, t)|^2 dx dt} Z_\Delta\,, \tag{5}$$

$$Z_\Delta = \int d[\bar{\psi}, \psi] \exp \left(- \iint \sum_{\alpha = \uparrow, \downarrow} \left(\frac{i}{2} \bar{\psi} \partial_t \psi + c.c. - \mathcal{H}_0(\bar{\psi}, \psi) - \bar{\Delta} \psi_\uparrow \psi_\downarrow - \Delta \bar{\psi}_\downarrow \bar{\psi}_\uparrow \right) d^3x dt \right)$$

Now one can formally integrate over ψ and $\bar{\psi}$. Show that the result is

$$Z = \int d[\bar{\Delta}, \Delta] e^{-\frac{1}{2\lambda}\iint |\Delta(\mathbf{x},t)|^2 dx dt} \det \begin{pmatrix} \partial_t - H_0 & \Delta(\mathbf{x},t) \\ \bar{\Delta}(\mathbf{x},t) & \partial_t + H_0 \end{pmatrix}, \quad H_0 = -\frac{1}{2}\nabla^2 - \mu \tag{6}$$

b) Consider the saddle point approximation for the functional integral (6). Show that the equation $\delta S/\delta\bar{\Delta} = 0$, for a constant Δ, has the form

$$\Delta = \lambda T \sum_{\mathbf{p},\omega_n} \frac{\Delta}{\xi_{\mathbf{p}}^2 + \omega_n^2 + |\Delta|^2}, \quad \omega_n = (2n+1)\pi T, \quad \xi_{\mathbf{p}} = \mathbf{p}^2/2m - \mu \tag{7}$$

Show that this equation is identical to the BCS gap equation.

c) The advantage of the functional integral approach is that it allows to consider spatially varying order parameter without changing the formalism. In the vicinity of the critical point, we expect Δ to be small. Expand the effective action obtained above,

$$S(\Delta) = \frac{1}{2\lambda}\iint |\Delta(\mathbf{x},t)|^2 d^3x dt - \ln\det \begin{pmatrix} i\partial_t - H_0 & \Delta(\mathbf{x},t) \\ \bar{\Delta}(\mathbf{x},t) & -i\partial_t + H_0 \end{pmatrix}, \tag{8}$$

in powers of spatially inhomogeneous Δ (assuming slow variation in space) and derive the Ginzburg-Landau functional[1]

$$S = \iint \left(a|\Delta|^2 + \frac{1}{2} b|\Delta|^4 + c|\nabla\Delta|^2 \right) d^3x dt \tag{9}$$

Relate the coefficients a, b, c to the parameters of the microscopic BCS hamiltonian. Show that the coefficient a of the quadratic term changes sign at the BCS transition point, while b and c are positive.

4. Path integral for spin $1/2$ (optional)

In this problem we first introduce a fermionic representation of the spin $1/2$ operators, and then use it to derive the Wess-Zumino path integral for spin.

a) Consider two fermions, described by the canonical operators $a_{1,2}$, $a_{1,2}^+$ and commutation relations $a_i a_j + a_j a_i = 0$, $a_i^+ a_j + a_j a_i^+ = \delta_{ij}$. Show that the operators

$$\sigma^+ = a_2^+ a_1, \quad \sigma^- = a_1^+ a_2, \quad \sigma_z = [\sigma^+, \sigma^-] = a_2^+ a_2 - a_1^+ a_1 \tag{10}$$

obey Pauli commutation relations. Note that, since these operators commute with the particle number operator $\hat{n} = a_1^+ a_1 + a_2^+ a_2$, the $2 \times 2 = 4$-dimensional Hilbert space has a 2-dimensional subspace, $\hat{n} = 1$, in which a canonical spin representation is realized.

b) Consider the Hamiltonian $\mathcal{H}_{\mathbf{n}} = \lambda \sum_{ij} a_i^+ a_j \sigma_{ij}^\alpha n_\alpha$, where $\mathbf{n}$ is a unit vector and σ_{ij}^α are Pauli matrices ($\alpha = x, y, z$). Starting from the reprentation of the partition function for $\mathcal{H}$ as an integral over *two* anticommuting variables, $Z = \int dx_1^* dx_2^* dx_1 dx_2 e^{i\int \left(ix_j^* \partial_t x_j - \mathcal{H}_{\mathbf{n}}\right) dt}$, perform gaussian integration over x_i and obtain an effective action for the dynanmcs of $\mathbf{n}$. Assuming slow time variation (for large λ), expand in the powers of the time derivative ∂_t to the lowest order. the result is the path integral for a unit vector $\mathbf{n}(t)$ that describes spin $1/2$.

c) Add Zeeman coupling to external magnetic field, $\mu\mathbf{B}\cdot\mathbf{n}$, and obtain Bloch equations as a saddle point approximation to this path integral.

[1]The virtue of the GL equation is that it allows to study fluctuations near the critical point, and, more importantly, the complex behavior of superconductors in magnetic field. The coupling to magnetic field is via the long derivative $-i\nabla \to -i\nabla - \frac{2e}{\hbar c}\mathbf{A}$.