Coherent states

1. Operator identities.

Here we prove two useful theorems from operator algebra that will be used in the problems of this homework and later in the course.

a) Let $\widehat{A}$ and $\widehat{B}$ be two operators that do not necessarily commute. Prove the so-called *operator expansion theorem*:

$$f(x) = \exp(x\widehat{A})\widehat{B}\exp(-x\widehat{A}) = \widehat{B} + x[\widehat{A}, \widehat{B}] + \frac{x^2}{2!}[\widehat{A}, [\widehat{A}, \widehat{B}]] + ... \tag{1}$$

with x a parameter. (Hint: consider the derivative $f'(x)$ and compare its Taylor series in x with that for $f(x)$.)

In the special case when the commutator $[\widehat{A}, \widehat{B}] = c$ is a c-number, the series terminates after the second term, giving

$$\exp(x\widehat{A})\widehat{B}\exp(-x\widehat{A}) = \widehat{B} + cx \tag{2}$$

Apply this result to the coordinate and momentum operators, $\widehat{B} = \widehat{q}$, $\widehat{A} = \widehat{p}/\hbar = -id/dq$.

b) Let A and B be two operators whose commutator $[\widehat{A}, \widehat{B}]$ commutes with both A and B (e.g., $[\widehat{A}, \widehat{B}] = c$ a c-number). Prove *the Campbell-Baker-Hausdorf theorem*:

$$\exp\left[x(\widehat{A} + \widehat{B})\right] = \exp\left(x\widehat{A}\right)\exp\left(x\widehat{B}\right)\exp\left(-x^2[\widehat{A}, \widehat{B}]/2\right) \tag{3}$$

For that, consider $\widehat{C}(x) = \exp\left(x\widehat{A}\right)\exp\left(x\widehat{B}\right)$, differentiate both sides with respect to x and, using the operator expansion theorem (1), show that $d\widehat{C}(x)/dx = \left(\widehat{A} + \widehat{B} + x[\widehat{A}, \widehat{B}]\right)\widehat{C}(x)$. Integrate with respect to x like an ordinary differential equation.

2. Displacement operators.

a) Consider the displacement operators, defined as

$$\widehat{D}(v) = \exp\left(v\widehat{a}^+ - \bar{v}\widehat{a}\right) \tag{4}$$

with v a complex parameter. Prove unitarity: $\widehat{D}^+(v)\widehat{D}(v) = 1$, $\widehat{D}^{-1}(v) = \widehat{D}(-v)$.

For a real-valued v show that in the q-representation the displacement operator (4) acts as an argument shift:

$$\widehat{D}(v)\psi(q) = \exp\left(\tilde{v}\partial_q\right)\psi(q) = \psi(q + \tilde{v}), \quad \tilde{v} = \sqrt{2}\lambda v \tag{5}$$

with the length $\lambda = \sqrt{\hbar/m\omega}$. (Hint: relate $\widehat{D}(v)$ to the Taylor series formula.)

b) Show that the coherent states can be obtained by "displacing" the vacuum state, $|v\rangle = \widehat{D}(v)|0\rangle$. (Use the operator expansion theorem (1)).

c) Show that the unitary transformation $\widehat{D}(v)$ displaces $\widehat{a}$ by v, and $\widehat{a}^+$ by $\bar{v}$,

$$\widehat{D}^+(v)\widehat{a}\widehat{D}(v) = \widehat{a} + v, \quad \widehat{D}^+(v)\widehat{a}^+\widehat{D}(v) = \widehat{a}^+ + \bar{v} \tag{6}$$

For any function of operators $\widehat{a}$ and $\widehat{a}^+$ with a power series expansion, show that

$$\widehat{D}^+(v) f(\widehat{a}, a^+) \widehat{D}(v) = f(\widehat{a} + v, a^+ + \bar{v}) \tag{7}$$

d) Prove the product formula

$$\widehat{D}(v') \widehat{D}(v) = e^{v'\bar{v} - \bar{v}' v} \widehat{D}(v + v') \tag{8}$$

Note that the displacement operators $\widehat{D}(v)$ and $\widehat{D}(v')$ commute only when $\arg(v) = \arg(v')$.

3. Harmonic oscillator excited by an external force.

a) Consider a particle moving in a parabolic potential in the presence of a time-dependent force, $\mathcal{H} = \frac{1}{2}\hbar\omega\,(\widehat{p}^2 + \widehat{q}^2) - F(t)\widehat{q}$. Show that the evolution in time of an arbitrary coherent state can be obtained using the displacement operators (4) studied in Problem 2. Assume that the evolved coherent state remains a coherent state at all times, so that

$$|\alpha\rangle(t) = \widehat{D}(v(t))|\alpha\rangle = |\alpha + v(t)\rangle \tag{9}$$

Obtain a differential equation for the function $v(t)$ and show that its real and imaginary parts correspond to the classical Hamilton equations $dq/dt = p$, $dp/dt = F(t)$.

Show that the unitary transformation $\mathcal{H}' = \widehat{D}(v(t))\mathcal{H}\widehat{D}^{-1}(v(t))$ gives a free oscillator Hamiltonian with $F(t) = 0$. It describes the transformation of the quantum problem to the classical co-moving reference frame.

How does the function $v(t)$ should evolve in time in order for is such that at all times it remains a coherent state

b) The harmonic oscillator of part a), initially in the ground state, was subject to a constant force during the time interval $0 < t < \tau$. Find the state at $t > \tau$. Determine the distribution of energies.

c) For the state found in part b) at $t > \tau$, find the phase-space density, i.e., the Wigner function $W(q, p)$, as a function of time.