1 Lecture 2. Squeezed States

In this lecture we shall continue the discussion of coherent states, focusing on their properties as a basis in Hilbert space. Also, we introduce the so-called squeezed states that minimize the uncertainty product $\delta q \delta p$. These states can be viewed as a generalization of the coherent states and, like the latter, are closely related to the properties of the operators a and a^+.

The coherent states discussed above are members of a wider class of states having the property that the product of the dispersions of $\hat{q}$ and $\hat{p}$ is a minimum. Such states are called 'squeezed states'. The possibility of reducing the uncertainty in a physical variable, e.g., coordinate or momentum of a mechanical oscillator, or the duration of an optical pulse, provided by the squeezed states, is often useful in applications.

1.1 Uncertainty product

Recall the proof of the uncertainty relation $\delta q \delta p \geq \frac{1}{2}\hbar$.

For any two hermitian operators A, B, and any state ψ, consider the quantity

$$F(\alpha) = \langle (\alpha\hat{A} + i\hat{B})\psi | (\alpha\hat{A} + i\hat{B})\psi \rangle = \langle \hat{A}^2 \rangle \alpha^2 + \langle i[\hat{A}, \hat{B}] \rangle \alpha + \langle \hat{B}^2 \rangle \tag{1}$$

where $\langle ... \rangle$ stands for the expectation value $\langle \psi | ... | \psi \rangle$. Since $F(\alpha)$ is nothing but the norm $||(\alpha\hat{A} + i\hat{B})\psi||^2$, it is non-negative. The quadratic polynomial $F(\alpha)$ thus does not have real roots, which gives

$$\langle \hat{A}^2 \rangle \langle \hat{B}^2 \rangle \geq \frac{1}{4} \left| \langle [\hat{A}, \hat{B}] \rangle \right|^2 \tag{2}$$

Applying this to $\hat{A} = \hat{q}$, $\hat{B} = \hat{p} = -i\hbar\partial_q$, obtain

$$F(\alpha) = \langle (\alpha\hat{q} + i\hat{p})\psi | (\alpha\hat{q} + i\hat{p})\psi \rangle = \langle \hat{q}^2 \rangle \alpha^2 + \langle i[\hat{q}, \hat{p}] \rangle \alpha + \langle \hat{p}^2 \rangle \geq 0 \tag{3}$$

leading to $\langle \hat{p}^2 \rangle \langle \hat{q}^2 \rangle \geq \frac{1}{4}\hbar^2$. For a state with $\bar{q} = \langle \hat{q} \rangle = 0$ and $\bar{p} = \langle \hat{p} \rangle = 0$, the coordinate and momentum uncertainty is

$$\langle \delta q^2 \rangle = \langle (\hat{q} - \bar{q})^2 \rangle = \langle \hat{q}^2 \rangle , \quad \langle \delta p^2 \rangle = \langle (\hat{p} - \bar{p})^2 \rangle = \langle \hat{p}^2 \rangle \tag{4}$$

which gives

$$\langle \delta q^2 \rangle \langle \delta p^2 \rangle \geq \frac{1}{4}\hbar^2 \tag{5}$$

The uncertainty relation for a more general state $\psi(q)$ with nonzero $\bar{q}$, $\bar{p}$ can be obtained from the above. Let us shift the coordinate and momentum as

$$\psi(q) = e^{i\bar{p}q}\tilde{\psi}(q - \bar{q}) \tag{6}$$

Since $\langle(\hat{q}-\bar{q})^2\rangle_\psi = \langle q^2\rangle_{\tilde{\psi}}$ and $\langle(\hat{p}-\bar{p})^2\rangle_\psi = \langle p^2\rangle_{\tilde{\psi}}$, one can write

$$\langle\delta q^2\rangle_\psi\langle\delta p^2\rangle_\psi = \langle q^2\rangle_{\tilde{\psi}}\langle p^2\rangle_{\tilde{\psi}} \geq \frac{1}{4}\hbar^2 \tag{7}$$

the uncertainty relation for an arbitrary state.

The states that minimize the uncertainty product and have $\bar{q}=\bar{p}=0$ satisfy

$$(\alpha\hat{q}+i\hat{p})\psi = \alpha q\psi + \hbar\psi' = 0 \tag{8}$$

where α is a complex parameter. Integrating this equation, with $\hbar$ absorbed in α, obtain

$$\psi(q) = \pi^{-1/4}(\alpha')^{1/2}\exp(-\alpha q^2/2)\,, \quad \alpha = \alpha' + i\alpha'' \tag{9}$$

(normalizability requires $\alpha' > 0$).

More general states with minimal uncertainty can be obtained by a displacement of p and q, as in Eq.(6). The general states that minimize $\langle\delta q^2\rangle\langle\delta p^2\rangle$ are called *squeezed states*. As we shall see, a squeezed state may be constructed to have and arbitrarily small width $\langle\delta q^2\rangle^{1/2}$.

1.2 Squeezed states and the operators a and a^+.

Suppose we prepared a squeezed state (9) of a harmonic oscillator $\mathcal{H} = \hbar\omega(a^+a+\frac{1}{2})$ with $\bar{q}=\bar{p}=0$, and ask how will it evolve in time. The simplest way to obtain the time evolution is to rewrite the minimal uncertainty condition $\alpha q\psi + \hbar\psi' = 0$ through the canonical creation and annihilation operators, since the latter evolve in time in a very simple way: $a(t) = e^{i\mathcal{H}t}a\,e^{-i\mathcal{H}t} = e^{-i\omega t}a$, $a^+(t) = e^{i\mathcal{H}t}a^+\,e^{-i\mathcal{H}t} = e^{i\omega t}a^+$.

Using the relations $q = \frac{\lambda}{\sqrt{2}}(a+a^+)$, $p = i\frac{\hbar}{\sqrt{2}\lambda}(a^+-a)$ (Lecture 1), we obtain

$$(\alpha\hat{q}+i\hat{p})\psi = \frac{1}{2\lambda}\left((\alpha\lambda^2-1)a + (\alpha\lambda^2+1)a^+\right)\psi = 0 \tag{10}$$

The time dependent state $\psi(t) = e^{-i\mathcal{H}t}\psi$ satisfies

$$e^{-i\mathcal{H}t}\left((\alpha\lambda^2-1)a + (\alpha\lambda^2+1)a^+\right)e^{i\mathcal{H}t}\psi(t) = \left((\alpha\lambda^2-1)a(-t) + (\alpha\lambda^2+1)a^+(-t)\right)\psi(t) = 0$$
$$\left((\alpha\lambda^2-1)e^{i\omega t}a + (\alpha\lambda^2+1)e^{-i\omega t}a^+\right)\psi(t) = 0 \tag{11}$$

This equation has the form of Eq.(10) with a time-dependent α given by

$$\frac{\lambda^2\alpha(t)-1}{\lambda^2\alpha(t)+1} = \frac{\lambda^2\alpha-1}{\lambda^2\alpha+1}\,e^{2i\omega t}\,, \quad \alpha(t) = \frac{\lambda^2\alpha\cos\omega t - i\sin\omega t}{\lambda^2\left(\cos\omega t - i\lambda^2\alpha\sin\omega t\right)} \tag{12}$$

The wavepacket remains gaussian at all times, while its width oscillates. For the ground state of the oscillator, $\alpha(t)\lambda^2 = 1$ at all times. If the initial state is squeezed, $\alpha\lambda^2 \gg 1$, the wavepacket width reaches maximum at the times when $\cos\omega t$ is close to zero, and collapses to the minimal value when $\sin\omega t$ is near zero.

The Wigner function of the squeezed state (9) evolving in time according to (12) is a gaussian distribution in the phase space, centered at the origin and rotating with the frequency ω without changing shape (see Problem 3, PS#2). In contrast with the isotropic phase-space distribution of the ground state (as well as any other coherent state), the squeezed states produce gaussians elongated in one direction and squashed in a perpendicular direction. The major axes rotate according to the classical oscillator phase flow.

There is another, more formal, definition of squeezed states based on the so-called 'squeeze operators'. These are unitary operators which, when applied to the oscillator vacuum state, produce a squeezed state. The simplest example of a squeeze operator is

$$U(\theta) = \exp\left(\theta\left(\hat{a}\hat{a} - \hat{a}^{+}\hat{a}^{+}\right)/2\right) \tag{13}$$

This operator has the following properties (see Problem 1, PS#2)

$$U^{+}(\theta)\hat{a}U(\theta) = \cosh\theta\hat{a} - \sinh\theta\hat{a}^{+}\,, \quad U^{+}(\theta)a^{+}U(\theta) = \cosh\theta\hat{a}^{+} - \sinh\theta\hat{a} \tag{14}$$

from which it follows that

$$U^{+}(\theta)\hat{q}U(\theta) = U^{+}(\theta)\frac{\lambda}{\sqrt{2}}\left(\hat{a} + \hat{a}^{+}\right)U(\theta) = e^{-\theta}\hat{q} \tag{15}$$

$$U^{+}(\theta)\hat{p}U(\theta) = U^{+}(\theta)\frac{i\hbar}{\sqrt{2}\lambda}\left(\hat{a}^{+} - \hat{a}\right)U(\theta) = e^{\theta}\hat{p} \tag{16}$$

Using these results, one can show that the state $U(\theta)|0\rangle$ is a minimum uncertainty state (9) with the parameter $\alpha\lambda^2 = e^{2\theta}$. Hence the width of the wavepacket $\langle\delta q^2\rangle^{1/2}$ is e^{θ} times smaller than that of oscillator vacuum.

Other squeezed states can be obtained in a similar way using unitary operators $U(z) = \exp\left(z\left(\hat{a}\hat{a} - \hat{a}^{+}\hat{a}^{+}\right)/2\right)$ with complex parameter z. Because of the form of the squeeze operator, the squeezed states are also sometimes called 'two-photon coherent states'.

1.3 Squeezed states from time evolution

How can one obtain a squeezed state, starting from, e.g., the vacuum state? It turns out that the squeezed states arise quite naturally from oscillator dynamics, provided that the parameters of the oscillator Hamiltonian, the frequency ω and the mass m, are functions of time. Let us consider the Schrödinger evolution

$$i\hbar\partial_t\psi = \mathcal{H}(t)\psi\,, \quad \mathcal{H}(t) = \frac{p^2}{2m(t)} + \frac{m(t)\omega^2(t)}{2}q^2 \tag{17}$$

Integrating formally the evolution equation in the time interval $[0, t]$, have

$$\psi(t) = \hat{S}(t)\psi_{t=0} = \lim_{N\to\infty}\prod_{j=1}^{N} e^{-\frac{i}{\hbar}\mathcal{H}(t_j)\Delta t}\psi_{t=0} \tag{18}$$

with $t_j = j\Delta t$, $\Delta t = t/N$.

We now show that the evolving squeezed state satisfies

$$(P(t)\hat{q} - Q(t)\hat{p})\,\psi(t) = 0 \tag{19}$$

where $P(t)$ and $Q(t)$ are functions of time to be found.

Indeed, let us define the quantity

$$\hat{C}(t) = \hat{S}(t)\,(P(0)\hat{q} - Q(0)\hat{p})\,\hat{S}^{-1}(t) \tag{20}$$

Then, at all times,

$$\hat{C}(t)\psi(t) = \hat{S}(t)\,(P(0)\hat{q} - Q(0)\hat{p})\,\hat{S}^{-1}(t)\hat{S}(t)\psi = 0 \tag{21}$$

The quantity $\hat{C}(t)$ obeys equation of motion of the form

$$\hbar\partial_t\hat{C}(t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(e^{-\frac{i}{\hbar}\mathcal{H}(t)\Delta t}\hat{C}(t)e^{\frac{i}{\hbar}\mathcal{H}(t)\Delta t} - \hat{C}(t)\right) = i[\hat{C}(t),\mathcal{H}(t)] \tag{22}$$

We also note that, if $\hat{C}$ is a polynomial in q, p of a degree n, then the commutator $[\hat{C},\mathcal{H}]$ is also a polynomial of exactly the same degree. Since $[\hat{C},\mathcal{H}]$ determines the rate of change of $\hat{C}$ with time, and the $\hat{C}$ starts as a linear function in q and p at $t = 0$, it is natural to suppose that $\hat{C}$ remains linear in in q and p at all times, $\hat{C}(t) = P(t)\hat{q} - Q(t)\hat{p}$. Using the evolution equation, we find

$$\hbar\partial_t\hat{C}(t) = i[\hat{C}(t),\mathcal{H}(t)] = i\Big[\,(P\hat{q} - Q\hat{p})\,,\left(\frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{q}^2\right)\Big] = -\hbar\left(\frac{P}{m}\hat{p} + m\omega^2 Q\hat{q}\right) \tag{23}$$

Substituting $\partial_t\hat{C}(t) = \dot{P}(t)\hat{q} - \dot{Q}(t)\hat{p}$, obtain

$$\dot{Q} = P/m\,,\quad \dot{P} = -m\omega^2 Q \tag{24}$$

which coincides with the classical Hamiltonian equations for the oscillator (17). The initial conditions, e.g., corresponding to the state (9), are generally speaking complex valued. For instance, for the oscillator vacuum state, have

$$P_{t=0} = 1\,,\quad Q_{t=0} = -i\lambda^2 \tag{25}$$

Thus the evolving state has the form (9) at all times with $\alpha(t) = -iP(t)/Q(t)$.

Let us consider **a simple example**, when the oscillator frequency jumps from ω_0 to ω. After that, the former ground state, which is not a ground state any more, starts to evolve in time. To analyze the dynamics of Q and P, it is convenient to introduce the variables $z_\pm = Q \pm iP/m\omega$ which obey

$$\dot{z}_+ = -i\omega z_+\,,\quad \dot{z}_- = i\omega z_- \tag{26}$$

Solving for the time evolution $z_+(t)$, obtain

$$z_+(t) = e^{-i\omega t}z_+(t=0) = e^{-i\omega t}\left(\frac{i}{m\omega} - \frac{i}{m\omega_0}\right) \tag{27}$$

Similarly, for $z_-(t)$, obtain

$$\tilde{z}_-(t) = e^{i\omega t}\tilde{z}_-(t=0) = -e^{i\omega t}\left(\frac{i}{m\omega} + \frac{i}{m\omega_0}\right) \tag{28}$$

which gives

$$Q(t) = \frac{1}{2}\left(z_+ + z_-\right) = \frac{\sin\omega t}{m\omega} - i\frac{\cos\omega t}{m\omega_0}, \quad P(t) = i\left(\frac{\sin\omega t}{m\omega_0} - i\frac{\cos\omega t}{m\omega}\right) \tag{29}$$

The resulting time dependence $\alpha(t) = -iP(t)/Q(t)$ agrees with Eq.(12).

As this example demonstrates, any time dependence of the Hamiltonian can be used to generate squeezed states. The underlying physical reason for this general behavior is that the certainty product, proportional to the phase space area of the Wigner density peak, remains constant in time due to the phase volume conservation in Hamiltonian dynamics.

However, in practice it is often desirable to get a highly squeezed state by employing only small variations of the oscillator parameters. The above example suggests that this may be difficult, since for the value ω close to ω_0 the wavepacket width varies very little as a function of time. However, large squeezing by small perturbation can be achieved by using the phenomenon of *parametric resonance*. It is well known (to any child on a swing) that, when the frequency of the oscillator is modulated,

$$\omega^2(t) = \omega_0^2 + \lambda\cos\Omega t \tag{30}$$

and the external frequency Ω is close to $2\omega_0$, the classical oscillator becomes unstable, with the oscillations of small amplitude growing larger as a function of time. A highly squeezed state is formed near the moments corresponding to the extremal points of classical motion, when $P(t)$ is large and $Q(t)$ small (being complex, $Q(t)$ never completely vanishes, but can become really small at large times). For further discussion of the details of parametric resonance we refer to Problem 2, PS#2 (see also Landau & Lifshits, "Classical Mechanics").