1 Lecture 1. Coherent States.

We start the course with the discussion of coherent states. These states are of interest because they provide

- a method to describe on equal terms both particles and photons;
- a connection to classical physics (mechanics and electrodynamics);
- tools for the construction of path integral, to be discussed later

The coherent states also provide a natural entry point into the method of second quantization that will be introduced in the next lecture.

1.1 Harmonic oscillator; the creation and annihilation operators

Particle in a parabolic potential:

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2\,, \quad p = -i\hbar\partial_q\,, \quad [q,p] = i\hbar \tag{1}$$

The ground state width can be found by minimizing energy:

$$\langle \mathcal{H} \rangle = \frac{\hbar^2}{2m\lambda^2} + \frac{m\omega^2\lambda^2}{2} \;\to\; \min \tag{2}$$

which gives $\lambda = (\hbar/m\omega)^{1/2}$.

It will be convenient to use nondimensionalized variables $q = \lambda\tilde{q}$, $p = (\hbar/\lambda)\tilde{p}$, so that the classical phase volume is rescaled by $\hbar$. Thus we obtain

$$\mathcal{H} = \frac{\hbar\omega}{2}\left(\tilde{p}^2 + \tilde{q}^2\right)\,, \quad \tilde{p} = -i\partial_{\tilde{q}}\,, \quad [\tilde{q},\tilde{p}] = i \tag{3}$$

We shall study the Hamiltonian (3) below having in mind the quantum-mechanical particle problem. However, later we shall find that the quantized electromagnetic field is also described by a set of harmonic oscillators of the form (3).

The canonical creation and annihilation operators are defined as

$$a = \frac{1}{\sqrt{2}}\left(\tilde{q} + i\tilde{p}\right)\,, \quad a^+ = \frac{1}{\sqrt{2}}\left(\tilde{q} - i\tilde{p}\right) \tag{4}$$

They can be used to express q, p and $\mathcal{H}$ as follows:

$$q = \frac{\lambda}{\sqrt{2}}\left(a + a^+\right)\,, \quad p = i\frac{\hbar}{\sqrt{2}\lambda}\left(a^+ - a\right) \tag{5}$$

$$\mathcal{H} = \frac{\hbar\omega}{2}\left(a^+a + aa^+\right) = \hbar\omega\left(a^+a + \frac{1}{2}\right) \tag{6}$$

The operators a and a^+ obey the commutation relation

$$[a, a^+] = 1 \tag{7}$$

Proof:

$$aa^+ - a^+a = \frac{1}{2}\left((\tilde{q} + i\tilde{p})(\tilde{q} - i\tilde{p}) - (\tilde{q} - i\tilde{p})(\tilde{q} + i\tilde{p})\right) = i\left(\tilde{p}\tilde{q} - \tilde{q}\tilde{p}\right) = 1 \tag{8}$$

As a simple application of the operators a and a^+, let us reconstruct the main facts of the harmonic oscillator quantum mechanics.

1. The ground state $|\psi_0\rangle$, also called *vacuum state*, provides the lowest possible energy expectation value

$$\langle\psi_0|\mathcal{H}|\psi_0\rangle = \hbar\omega\left(\langle\psi_0|a^+a|\psi_0\rangle + \frac{1}{2}\right) = \hbar\omega\left(\langle a\psi_0|a\psi_0\rangle + \frac{1}{2}\right) \tag{9}$$

which gives the condition $a\psi_0 = 0$, i.e., $(q + ip)\psi_0 = 0$.

Let us find the ground (vacuum) state in the q-representation. Using the units with the length $\lambda = 1$, i.e., $\tilde{q}$, $\tilde{p}$ instead of q, p, we write

$$q\psi_0(q) + \psi_0'(q) = 0\,, \quad \psi_0'/\psi_0 = -q\,, \quad \ln\psi_0 = -q^2/2 \tag{10}$$

This leads to a Gaussian wavefunction

$$\psi_0(q) = \pi^{-1/4}\exp\left(-q^2/2\right)\,, \quad E_0 = \hbar\omega/2 \tag{11}$$

2. The higher energy states can be obtained from the ground state. Starting with the commutation relations,

$$a^+\mathcal{H} = (\mathcal{H} - \hbar\omega)\,a^+\,, \quad a\mathcal{H} = (\mathcal{H} + \hbar\omega)\,a \tag{12}$$

one can show that the states $\psi_n(q) = (a^+)^n\psi_0(q)$ are the eigenstates. Indeed, consider $\psi_1 = a^+\psi_0$ and apply the first relation (12):

$$(\mathcal{H} - \hbar\omega)\,\psi_1 = a^+\mathcal{H}\psi_0 = E_0 a^+\psi_0 = E_0\psi_1 \tag{13}$$

which gives $E_1 = E_0 + \hbar\omega = 3\hbar\omega/2$ and

$$\psi_1(q) = \frac{1}{\sqrt{2\hbar}}\left(q - \partial_q\right)\psi_0 = \frac{2}{\sqrt{2\hbar}}\,q\exp\left(-q^2/2\right) \tag{14}$$

Subsequently, from ψ_1 one obtains the eigenstate $\psi_2(q) \propto (2q^2 - 1)\exp(-q^2/2)$ with the energy $E_2 = E_1 + \hbar\omega = 5\hbar\omega/2$, and so on. The recursion relation $\psi_n = a^+\psi_{n-1}$, $E_n = E_{n-1} + \hbar\omega$, gives

$$\psi_n = A_n\,(a^+)^n\psi_0\,, \quad E_n = \hbar\omega\left(n + \frac{1}{2}\right) \tag{15}$$

where we inserted the normalization factors A_n.

The factors A_n can be determined from

$$1 = A_n^2 \langle (a^+)^n \psi_0 | (a^+)^n \psi_0 \rangle = A_n^2 \langle \psi_0 | a...aa^+...a^+ | \psi_0 \rangle \tag{16}$$

$$= A_n^2 \langle \psi_0 | a^{n-1} \left(\frac{\mathcal{H}}{\hbar\omega} + \frac{1}{2} \right) (a^+)^{n-1} | \psi_0 \rangle \tag{17}$$

$$= A_n^2 n \langle \psi_0 | a^{n-1} (a^+)^{n-1} | \psi_0 \rangle = ... = A_n n! \langle \psi_0 | \psi_0 \rangle = A_n n! \tag{18}$$

which gives $A_n = (n!)^{-1/2}$. The normalized oscillator eigenstates

$$|n\rangle = \frac{1}{\sqrt{n!}} (a^+)^n |0\rangle \,, \quad |0\rangle = \psi_0 \tag{19}$$

form an orthonormal complete set of functions, providing a basis in the oscillator Hilbert space. The ground state $|0\rangle$ is also known as *the vacuum state.*

3. The operators a and a^+ written as matrices in the basis of states (19) have nonzero matrix elements only between the states $|n\rangle$ and $|n \pm 1\rangle$:

$$\langle n | a^+ | m \rangle = \sqrt{n} \delta_{n,m+1} \,, \quad \langle m | a | n \rangle = \sqrt{n} \delta_{n,m+1} \tag{20}$$

while all other matrix elements are zero.

4. It is convenient to define the so-called *number operator* $\hat{n} = a^+ a$ which counts the number of energy quanta in the QM particle problem, or the number of photons for quantized E&M field. In the energy basis $|n\rangle$, the number operator is diagonal:

$$\hat{n} |n\rangle = a^+ a |n\rangle = n |n\rangle \,, \quad \mathcal{H} = \hbar\omega \left(\hat{n} + \frac{1}{2} \right) \tag{21}$$

1.2 Definition of coherent states.

The coherent states are defined as eigenstates of the operator a:

$$a|v\rangle = v|v\rangle \tag{22}$$

where v is a complex parameter. Expanded in the energy basis (19), $|v\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$, the coherent state can be reconstructed from

$$a|v\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n} |n-1\rangle = \sum_{n=0}^{\infty} v c_n |n\rangle \tag{23}$$

Comparing the coefficients, obtain a recursion relation $c_n = (v/\sqrt{n}) c_{n-1}$, leading to

$$c_n = \frac{v^n}{\sqrt{n!}} c_0 \tag{24}$$

The coefficient c_0 is determined from normalization

$$1 = \sum_{n=0}^{\infty} |c_n|^2 = \sum_{n=0}^{\infty} \frac{|v|^{2n}}{n!} |c_0|^2 = e^{|v|^2} |c_0|^2 \tag{25}$$

Finally,

$$|v\rangle = e^{-|v|^2/2} \sum_{n=0}^{\infty} \frac{v^n}{\sqrt{n!}} |n\rangle = e^{-|v|^2/2} e^{va^+} |0\rangle \tag{26}$$

As an **example**, consider the distribution of the number of quanta $\hat{n} = a^+ a$ in a coherent state. Since $\hat{n}|n\rangle = n|n\rangle$, the distribution is given by

$$p_n = |c_n|^2 = e^{-|v|^2} \frac{|v|^{2n}}{n!} \tag{27}$$

This is a Poisson distribution with the mean $\bar{n} = |v|^2$.

1.3 The quasiclassical interpretation of coherent states

As we shall see below, the coherent states represent the points of the classical phase space (q, p). This can be conjectured most easily from their time dependence. Applying the Schrödinger equation $i\partial_t \psi = \mathcal{H}\psi$ to the number states, we have

$$|n\rangle(t) = e^{-i(n+\frac{1}{2})\omega t} |n\rangle \tag{28}$$

for the number states. Combined with (26), this gives

$$|v\rangle(t) = e^{-|v|^2/2} \sum_{n=0}^{\infty} \frac{v^n}{\sqrt{n!}} e^{-i(n+\frac{1}{2})\omega t} |n\rangle = e^{-i\omega t/2} |v(t)\rangle \tag{29}$$

with

$$v(t) = e^{-i\omega t} v \tag{30}$$

This defines a circular trajectory in the complex v plane, suggesting the correspondence with classical coordinate and momentum,

$$q = c\, v' \,, \quad p = c\, v'' \,, \quad v = v' + i v'' \tag{31}$$

where c is a scaling factor. The relation of coherent states with the points in a classical phase space will be clarified below.

Let us find the form of a coherent state in the q-representation, $\psi_v(q) = \langle q|v\rangle$. As before, we use the units in which the length $\lambda = 1$, and write

$$v\psi_v(q) = \langle q|a|v\rangle = \frac{1}{\sqrt{2}} \langle q| \left(q + \partial_q\right) |v\rangle = \frac{1}{\sqrt{2}} \left(q + \partial_q\right) \psi_v(q) \tag{32}$$

Solving the equation $q\psi + \psi' = \sqrt{2} v \psi$, obtain

$$\ln \psi = -q^2/2 + \tilde{v} q + \text{ const.} \,, \quad \tilde{v} = \sqrt{2} v \tag{33}$$

and, finally,

$$\psi_v(q) = A \exp\left[-\frac{1}{2} \left(q - \tilde{v}\right)^2 \right] \,, \quad |A| = \pi^{-1/4} e^{-(\tilde{v}'')^2/2} = \pi^{-1/4} e^{-(v'')^2} \tag{34}$$

with $\tilde{v}'' = \sqrt{2}v''$. The probability $|\psi_v(q)|^2$ has a form of a gaussian centered at $q = \mathrm{Re}(\tilde{v})$, which agrees with the above interpretation of v as a point in the phase space (with the scaling factor taking value $c = \sqrt{2}$).

A more detailed picture of the phase-space density is provided by the Wigner distribution function

$$W(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\langle q+\frac{1}{2}x|\hat{\rho}|q-\frac{1}{2}x\rangle e^{ixp/\hbar}dx \tag{35}$$

where $\hat{\rho}$ is the density matrix. For a pure state $\psi(q)$, the density matrix in position space is just $\hat{\rho}_{q,q'} = \bar{\psi}(q)\psi(q')$, and the matrix element in (35) is

$$\langle ...\rangle = \bar{\psi}(q-\frac{1}{2}x)\psi(q+\frac{1}{2}x) \tag{36}$$

The interpretation of the Wigner function as a phase-space density is supported by the following observations. One can check that the function (35) is real and normalized to unity. Also, the coordinate and momentum distributions, obtained by integrating over the conjugate variable, are reproduced correctly. The distribution in q is

$$\int W(q,p)dp = \langle q|\hat{\rho}|q\rangle \tag{37}$$

which is equal to $|\psi(q)|^2$ for a pure state, while the distribution in p is

$$\int W(q,p)dq = ... = \frac{1}{2\pi\hbar}|\psi(p)|^2 \tag{38}$$

where $\psi(p) = \int \psi(q)e^{-iqp}dq$.

For a coherent state $|v\rangle$, the Wigner function is given by

$$W(q,p) = \frac{|A|^2}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(q+\frac{1}{2}x-\bar{\tilde{v}})^2}e^{-\frac{1}{2}(q-\frac{1}{2}x-\tilde{v})^2}e^{ixp}dx \tag{39}$$

$$= \frac{|A|^2}{2\pi\hbar}\int_{-\infty}^{\infty} e^{-(q-\tilde{v}')^2}e^{ixp-\frac{1}{2}(\frac{1}{2}x+i\tilde{v}'')^2}dx = \frac{1}{2\pi\hbar}e^{-(q-v')^2-(p-\tilde{v}'')^2} \tag{40}$$

with $\tilde{v}' = \sqrt{2}v$, $\tilde{v}'' = \sqrt{2}v$. The gaussian distribution, centered at $q = \tilde{v}'$, $p = \tilde{v}''$, evolves in time as if carried by the classical harmonic oscillator phase flow. Since $v(t) = e^{-i\omega t}v$, the center of the gaussian packet is circling around the phase space origin:

$$W(q,p,t) = \frac{1}{2\pi\hbar}e^{-(q-|\tilde{v}|\cos\omega t)^2-(p-|\tilde{v}|\sin\omega t)^2} \tag{41}$$

For any $|v\rangle$, the width of the Wigner distribution is the same as for the vacuum state $|0\rangle$. Thus one can conclude that a coherent state can be thought of as a *displaced vacuum state*. This interpretation will be substantiated in Problem 2, PS#1.

1.4 Coherent states vector algebra

Here we discuss the the vector space propeties of coeherent states. Normally, the states appearing in quantum mechanics are orthogonal, or can be made orthogonal in some natural way, which provides an orthonormal basis in Hilbert space. The situation with coherent states is quite different.

Let us start with evaluating the overlap:

$$\langle u|v\rangle = e^{-\frac{1}{2}|u|^2} e^{-\frac{1}{2}|v|^2} \sum_{n=0}^{\infty} \frac{(\bar{u}v)^n}{n!} = e^{-\frac{1}{2}|u|^2-\frac{1}{2}|v|^2+\bar{u}v} \tag{42}$$

which shows that the coherent states are not orthogonal. On the other hand, Eq.(42) gives overlap decreasing exponentially as a function of the distance between u and v in the complex plane:

$$|\langle u|v\rangle|^2 = e^{-|u-v|^2} \tag{43}$$

For generic classical states, $|u|$, $|v| \gg 1$, the overlap is very small, which is consistent with the intuition that different classical states are orthogonal in the quantum mechanical sense.

Recalling the interpretation of the complex v plane as a phase space, $\tilde{q} = v'$, $\tilde{p} = v''$, we see that the overlap falls to zero at the length scale of the order of the wavepacket width set by Planck's constant, i.e. by the uncertainty relation, $\delta q \sim \lambda \propto \sqrt{\hbar}$, $\delta p \sim \hbar/\lambda \propto \sqrt{\hbar}$.

Another property of coherent states is completeness in the vector algebra sense. (A set of vectors is called complete if linear combinations of these vectors span the entire vector space.) The property is seen most readily from the formula know as unity decomposition:

$$\int |v\rangle\langle v| \frac{d^2v}{\pi} = \hat{1} \tag{44}$$

Proof can be obtained by evaluating the matrix elements of the operator on the left hand side of Eq. (44) between the number states

$$\Big\langle m \Big| \int |v\rangle\langle v| \frac{d^2v}{\pi} \Big| n \Big\rangle = \int e^{-|v|^2} \frac{\bar{v}^m v^n}{\sqrt{m!n!}} \frac{d^2v}{\pi} = \int_0^{\infty} \int_{-\pi}^{\pi} e^{-r^2} \frac{r^{m+n} e^{i(n-m)\theta}}{\sqrt{m!n!}} \frac{r dr d\theta}{\pi} \tag{45}$$

$$= \delta_{m,n} \int_0^{\infty} e^{-r^2} \frac{r^{2n}}{n!} dr^2 = \delta_{m,n} \tag{46}$$

(we used polar coordinates $v = re^{i\theta}$).

Using the formula (44), one can express any operator in terms of coherent states:

$$\hat{M} = \hat{1}\hat{M}\hat{1} = \iint |u\rangle\langle v| \, M(u,v) \frac{d^2v d^2u}{\pi^2} \tag{47}$$

with the matrix elements $M(u,v) = \langle u|\hat{M}|v\rangle$. This formula can be useful in calculations, as well as in formal manipulations (we shall use it later to derive Feynman path integral).

As another application of Eq. (44), let show that the coherent states form an overcomplete set, i.e. they are not linearly independent. Indeed, by writing

$$|v\rangle = \hat{1}|v\rangle = \int |u\rangle\langle u|v\rangle \frac{d^2u}{\pi} = \int |u\rangle \, e^{-\frac{1}{2}|u|^2-\frac{1}{2}|v|^2+\bar{u}v} \frac{d^2u}{\pi} = \int |u\rangle \, e^{\bar{u}v-\bar{v}u} e^{-\frac{1}{2}|u-v|^2} \frac{d^2u}{\pi} \tag{48}$$

we express the state $|v\rangle$ as a superposition of the states $|u\rangle$ with $|u-v| \leq 1$.

The overcompleteness (48) should not come as a surprise. The coherent states, parameterized by complex numbers, form a continuum, and thus there are way too many of them to form an a set of independent vectors. In contrast, the number states, which provide a basis of the oscillator Hilbert space, are a countable set.

To summarize, the coherent states are non-orthogonal and form an over-complete set. There have been many attempts to reduce the number of these states to a 'neccessary minimum,' by identifying a good subset that could serve as a basis. Even though some of the proposals are very interesting (e.g. Perelomov lattices[1]) it is probably more natural to use the entire space of coherent states, coping with the overcompleteness and not favoring some of the states to the others.

1.5 Coordinate and momentum uncertainty

We already mentioned, while discussing the Wigner function, that the coherent states form wavepackets in the phase space of width corresponding to the absolute minimum required by the uncertainty relation. Let us estimate coordinate uncertainty of a state $|u\rangle$:

$$\begin{aligned}
\langle u|\hat{q}^2|u\rangle &= \frac{\lambda^2}{2}\langle u|(a+a^+)^2|u\rangle = \frac{\lambda^2}{2}\langle u|a^2+a^{+2}+2a^+a+1|u\rangle = \frac{\lambda^2}{2}\left((u+\bar{u})^2+1\right) \\
(\langle u|\hat{q}|u\rangle)^2 &= \frac{\lambda^2}{2}\left(\langle u|a+a^+|u\rangle\right)^2 = \frac{\lambda^2}{2}(u+\bar{u})^2 \\
\langle u|\delta\hat{q}^2|u\rangle &= \langle u|\hat{q}^2|u\rangle - (\langle u|\hat{q}|u\rangle)^2 = \frac{\lambda^2}{2} = \frac{\hbar}{2m\omega} \qquad (49)
\end{aligned}$$

The uncertainty does not depend on u, which is consistent with the observations made using Wigner function. Similarly, for momentum uncertainty,

$$\begin{aligned}
\langle u|\hat{p}^2|u\rangle &= \frac{(i\hbar)^2}{2\lambda^2}\langle u|(a^+-a)^2|u\rangle = \frac{(i\hbar)^2}{2\lambda^2}\langle u|a^{+2}+a^2-2a^+a-1|u\rangle = \frac{\hbar^2}{2\lambda^2}\left(1-(u-\bar{u})^2\right) \\
(\langle u|\hat{p}|u\rangle)^2 &= \frac{(i\hbar)^2}{2\lambda^2}\left(\langle u|a^+-a|u\rangle\right)^2 = \frac{(i\hbar)^2}{2\lambda^2}(u-\bar{u})^2 \\
\langle u|\delta\hat{p}^2|u\rangle &= \langle u|\hat{p}^2|u\rangle - (\langle u|\hat{p}|u\rangle)^2 = \frac{\hbar^2}{2\lambda^2} = \frac{\hbar m\omega}{2} \qquad (50)
\end{aligned}$$

which is also independent of u. The uncertainty product $\langle\delta\hat{p}^2\rangle^{1/2}\langle\delta\hat{q}^2\rangle^{1/2}$ equals $\frac{1}{2}\hbar$, which is the lower bound required by the uncertainty relation. Below we shall see that coherent states can be naturally generalized to a broader class of states that minimize uncertainty product.

[1]One can consider lattices in the complex plane, $v_{m,n} = mu_1 + nu_2$, m, $n \in Z$. Perelomov shown that the lattice $\{v_{m,n}\}$ generates an undercomplete set of coherent states $\{|v_{m,n}\rangle\}$ if the area of the lattice unit cell is greater than $2\pi\hbar$, and an overcomplete set if the area is less than $2\pi\hbar$. The borderline lattices, having the unit cell area equal to $2\pi\hbar$, are overcomplete just by one vector. After any single vector is removed from such a lattice, it becomes a complete set.