Problem 1 (i) Incident wavepacket

\[ \psi_i(x, t) = \sum_k \psi_k(x, t) e^{ik(x - \xi_k t)} f_k = \sum_k e^{i\tilde{\xi}_k(x - \bar{x} t)} e^{-\frac{i}{2} \tilde{\xi}_k^2 t} f_k \]

(5 pts) We expanded \( \xi_k = \tilde{\xi} + \frac{d}{d\xi_k} \bar{x} \), \( \bar{x} = k - \bar{k} \).

Reflected wavepacket:

\[ \psi_o(x, t) = \sum_{k'} \int \psi_k(x, t) e^{i(kx - \xi_k t)} f_k \]

Expanding \( 2S(\xi_k) = 2S(\tilde{\xi}) + (\xi_k - \tilde{\xi}) \Delta \xi_k \), \( \Delta \xi_k = \frac{dS}{d\xi_k} \)

thus \( \Delta \xi_k = 2 \frac{dS}{d\xi_k} \) is a delay time.

For BW model \( \tilde{S}(\xi) = \tan^{-1} \frac{\xi}{2(\xi - \xi_0)} \) gives

\[ \Delta \xi_k = \frac{1}{\xi_k} \]

(5 pts) For \( \psi_i = \sum_k e^{ikx - i\xi_k t} f_k \)

\( \Delta \xi_{max} = \frac{1}{\Gamma} \)

and \( \psi_o = \sum_k \frac{\xi_k - \bar{\xi}}{\xi_k - \bar{\xi}^*} e^{i(kx - \xi_k t)} f_k \), \( \bar{\xi} = \xi_0 + \frac{i}{2} \Gamma \)

Writing \( \frac{\xi_k - \bar{\xi}}{\xi_k - \bar{\xi}^*} \) in the time representation as

\[ \frac{\xi_k - \bar{\xi}}{\xi_k - \bar{\xi}^*} = \int [S(\xi') + \Theta(t') e^{\frac{i}{2} \xi_k t'} e^{-\frac{i}{2} \xi_0 t'}] e^{-\frac{i}{2} \xi_0 t'} dt' \]

we can bring \( \psi_o \) to the form

\[ \psi_o = \sum_k e^{i(kx - \xi_k t)} f_k + \int dt \Gamma e^{\frac{i}{2} \xi_k t} e^{-\frac{i}{2} \xi_0 t'} \]

This is a sum of a delayed and non-delayed parts. The probability to be delayed by \( t \) is \( P(t) \propto e^{-\frac{t}{\Gamma}} \) accounts for the possibility that particle can spend some time on the resonant level.
Problem 2

(i) (4 points)

Let us choose basis $|11\rangle, |12\rangle, \ldots, |MN\rangle, |12N\rangle$, where $|ai\rangle$ denotes $i$-th channel in the $a$-th reservoir ($i=1..N$, $a=1,2$). Then the scattering matrix is

$$S = \begin{pmatrix} r_{11} & t_{12} \\ t_{12} & r_{22} \end{pmatrix}$$

Contribution of states in the energy interval $d\epsilon$ from $i$-th channel to the current $1 \rightarrow 2$ is given by:

$$dI_{1 \rightarrow 2,i} = \frac{e}{h} \frac{d\epsilon}{2\pi} \left( 1 - f_1(\epsilon) \right) \left[ f_1(\epsilon) \sum_{j=1}^{N} \left| t_{12} t_{12}^+ \right|^2 \right]$$

Similar expression is valid for $dI_{2 \rightarrow 1,i}$. Summing over channels and integrating over $d\epsilon$, we get:

$$I = \int dI_{1 \rightarrow 2} - dI_{2 \rightarrow 1} = \frac{e}{h} \int \frac{d\epsilon}{2\pi} \left[ f_1(\epsilon) \text{Tr} \left( t_{12} t_{12}^+ \right) - f_2(\epsilon) \text{Tr} \left( t_{21} t_{21}^+ \right) \right]$$

(ii) (6 points)

$$S = \begin{pmatrix} r_{11} & t_{12}^+ \\ t_{21}^+ & r_{22}^+ \end{pmatrix}$$

$$SS^+ = \begin{pmatrix} r_{11} t_{12}^* + t_{21} t_{21}^* & r_{11} t_{12} + t_{21} r_{22}^* \\ t_{12}^* t_{12} + t_{21} t_{22}^* & r_{22} t_{12}^* + r_{22} r_{22}^* \end{pmatrix} = I$$
Taking trace of $1\times1$ block, we obtain:

$$\text{Tr} \left( r_{i1} r_{i1}^* + t_{21} t_{21}^* \right) = N$$

Similarly, from $S^* S = \mathbb{1}$ we get:

$$\text{Tr} \left( r_{i1} r_{i1}^* + t_{12} t_{12}^* \right) = N$$

Therefore, $\text{Tr} \left( t_{12} t_{12}^* \right) = \text{Tr} \left( t_{21} t_{21}^* \right)$

$$\mu_2 = \mu_2 + eV \Rightarrow$$

$$I = \frac{e}{\hbar} \int \frac{d\epsilon}{2\pi} \left[ f_1(\epsilon) - f_2(\epsilon) \right] \text{Tr} \left( t_{21} t_{21}^* \right) =$$

$$= \frac{e}{\hbar} \int \frac{d\epsilon}{2\pi} \text{Tr} \left( t_{12} t_{12}^* \right).$$

For $V \to 0$, $S$-matrix can be treated as constant, and we get:

$$G = \frac{e^2}{2\pi \hbar} \text{Tr} \left( t_{21} t_{21}^* \right)$$

**Note:** $G_{\text{min}} = 0$, $G_{\text{max}} = N \frac{e^2}{\hbar}$
Problem 3

(i) Transition rate between plane wave states (Fermi's Golden Rule)

\[ W(p', p) = 2\pi |V_{p'p}|^2 \delta(\varepsilon_p - \varepsilon_{p'}) \]

For s-function potential

\[ V = \frac{dN}{d\varepsilon} \]

\[ \tau_{tr} = \sum_{p'} (1 - \cos \theta_{pp'}) W(p', p) = 2\pi V \hbar^2 n_{imp} \]

\[ \tau = \frac{V}{2\pi V (W^2 n_{imp})} \sigma = \frac{e^2}{\hbar} \frac{k_F \epsilon}{2} = \frac{n e^2}{m} \tau_{tr} \]

(ii) \( \rho_{xx} = \frac{m}{n e^2} \frac{1}{\tau_{tr}} \) (no dependence on B)

(iii) \( \rho_{xy} = \frac{eB}{\hbar} \) see lectures