8.513 HW #3

Problem 1 (i) Incident wavepacket

$$\psi_{in}(x,t) = \sum_k e^{i(kx-\varepsilon_k t)} f_k \approx \sum_k e^{i\tilde{k}(x-v_g t)} e^{-i\bar{\varepsilon}t} f_k$$

(5pts) we expanded $\varepsilon_k = \bar{\varepsilon} + \frac{d\varepsilon}{dk}\tilde{k}$, $\tilde{k} = k - \bar{k}$

Reflected wavepacket: $v_g = d\varepsilon/dk\,|_{k=\bar{k}}$

$$\psi_{out}(x,t) = \sum_k e^{2i\delta(\varepsilon_k)} e^{i(kx-\varepsilon_k t)} f_k$$

Expanding $2\delta(\varepsilon_k) = 2\delta(\bar{\varepsilon}) + (\varepsilon_k - \bar{\varepsilon})\Delta\tau$, $\Delta\tau = 2\frac{d\delta(\varepsilon)}{d\varepsilon}$

find $\psi_{out}(x,t) \approx e^{i\theta} \sum_k e^{i(kx-\varepsilon_k(t-\Delta\tau))} f_k$

Thus $\Delta\tau = 2\, d\delta/d\varepsilon$ is a delay time

For BW model $\delta(\varepsilon) = \tan^{-1}\frac{\gamma}{2(\varepsilon-\varepsilon_0)}$ gives $\Delta\tau = \frac{\Gamma}{(\varepsilon-\varepsilon_0)^2+\Gamma^2/4}$

(ii) For $\psi_{in} = \sum_k e^{ikx-i\varepsilon_k t} f_k$ $\qquad \Delta\tau_{max} = 4/\Gamma$

(5pts) and $\psi_{out} = \sum_k \frac{\varepsilon_k - z}{\varepsilon_k - z^*} e^{i(kx-\varepsilon_k t)} f_k$, $z = \varepsilon_0 + \frac{i}{2}\Gamma$

Writing $\frac{\varepsilon_k - z}{\varepsilon_k - z^*}$ in the time representation as

$$\frac{\varepsilon_k - z}{\varepsilon_k - z^*} = \int \left[\delta(t') + \theta(t') e^{-\frac{\Gamma}{2}t'} e^{-i\varepsilon_0 t'}\right] e^{i\varepsilon_k t'} dt'$$

we can bring ψ_{out} to the form

$$\psi_{out} = \sum_k e^{i(kx-\varepsilon_k t)} f_k + \int_0^\infty dt\, \Gamma e^{-\frac{\Gamma t'}{2}} e^{-i\varepsilon_0 t'} \times \sum_k e^{ikx-i\varepsilon_k t} f_k$$

This is a sum of a delayed and non-delayed parts. The probability to be delayed by $t' = \tau$ is $P(\tau) \propto e^{-\Gamma\tau}$ accounts for the possibility that particle can spend some time on the resonant level

Problem 2

(i) $\boxed{\text{(4 points)}}$

Let us choose basis $|11\rangle, |12\rangle \ldots, |1N\rangle, |21\rangle \ldots |2N\rangle$, where $|\alpha i\rangle$ denotes i-th channel in the αth reservoir ($i = 1..N$, $\alpha = 1,2$). Then the scattering matrix is

$$S = \begin{pmatrix} r_{11} & t_{21} \\ t_{12} & r_{22} \end{pmatrix}$$

Contribution of states in the energy interval $d\varepsilon$ from ith channel to the current $1 \to 2$ is given by:

probability to be unoccupied and occupied

$$dI_{1\to 2, i} = \frac{e}{\hbar} \frac{d\varepsilon}{2\pi} \left(1 - f_2(\varepsilon)\right) f_1(\epsilon) \cdot \sum_{j=1}^{N} |t_{12\, ij}|^2$$

Prob. to be transmitted

Similar expression is valid for $dI_{2\to 1, i}$.

Summing over channels and integrating over $d\varepsilon$, we get:

$$I = \int dI_{1\to 2} - dI_{2\to 1} = \frac{e}{\hbar} \int \frac{d\epsilon}{2\pi} \left[f_1(\epsilon) \operatorname{Tr}(t_{12} t_{12}^{+}) - f_2(\epsilon) \operatorname{Tr}(t_{21} t_{21}^{+}) \right]$$

(ii) $\boxed{\text{(6 points)}}$

$$S^{+} = \begin{pmatrix} r_{11}^{+} & t_{12}^{+} \\ t_{21}^{+} & r_{22}^{+} \end{pmatrix}$$

$$SS^{+} = \begin{pmatrix} r_{11} r_{11}^{+} + t_{21} t_{21}^{+} & r_{11} t_{12}^{+} + t_{21} r_{22}^{+} \\ t_{12} r_{11}^{+} + r_{22} t_{21}^{+} & t_{12} t_{12}^{+} + r_{22} r_{22}^{+} \end{pmatrix} = \hat{1}$$

Taking trace of 1-1 block, we obtain:

$$\mathrm{Tr}\,(r_{11} r_{11}^{+} + t_{21} t_{21}^{+}) = N$$

Similarly, from $S^{+} S = \hat{1}$ we get

$$\mathrm{Tr}\,(r_{11} r_{11}^{+} + t_{12} t_{12}^{+}) = N$$

Therefore, $\underline{\mathrm{Tr}\,(t_{12} t_{12}^{+}) = \mathrm{Tr}\,(t_{21} t_{21}^{+})}$

$\mu_1 = \mu_2 + eV \Rightarrow$

$$I = \frac{e}{\hbar} \int \frac{d\epsilon}{2\pi} [f_1(\epsilon) - f_2(\epsilon)]\, \mathrm{Tr}\,(t_{21} t_{21}^{+}) =$$

$$= \frac{e}{\hbar} \int_0^V \frac{d\epsilon}{2\pi} \cdot \mathrm{Tr}\,(t_{21} t_{21}^{+}) .$$

For $V \to 0$, S-matrix can be treated as constant, and we get:

$$\boxed{G = \frac{e^2}{2\pi\hbar} \mathrm{Tr}\,(t_{21} t_{21}^{+})}$$

Note: $G_{min} = 0$, $G_{max} = N \frac{e^2}{h}$

Problem 3

(i) Transition rate between plane wave States (Fermi's Golden Rule)

(5pts)

$$W(p',p) = 2\pi |V_{p'-p}|^2 \, \delta(\varepsilon_p - \varepsilon_{p'})$$

For δ-function potential $\nu = \frac{dN}{d\varepsilon}$

$$\tau_{tr}^{-1} = \sum_{p'} (1-\cos\theta_{pp'}) W(p',p) = 2\pi \nu |u|^2 n_{imp}$$

$$\ell = v_F \tau_{tr} = \frac{v_F}{2\pi\nu |u|^2 n_{imp}}, \quad \sigma = \frac{e^2}{h}\frac{k_F \ell}{2} = \frac{ne^2}{m}\tau_{tr}$$

(ii) $\rho_{xx} = \frac{m}{ne^2}\frac{1}{\tau_{tr}}$ (no dependence on B)

(5pts) $\rho_{xy} = \frac{eB}{n}$ see lectures