8.513 HW #2

Problem 1 (i) (5 pts)

$$\varepsilon\psi = -\frac{\hbar^2}{2m}\psi_{xx} - \frac{1}{2}ax^2\psi + U_0\psi - \frac{\hbar^2}{2m}\psi_{yy} + \frac{1}{2}by^2\psi$$

$$\psi(x,y) = \sum_n \varphi_n(y)\,\psi_n(x) \qquad \varepsilon_n = \hbar\sqrt{\tfrac{b}{m}}\left(n+\tfrac{1}{2}\right)$$

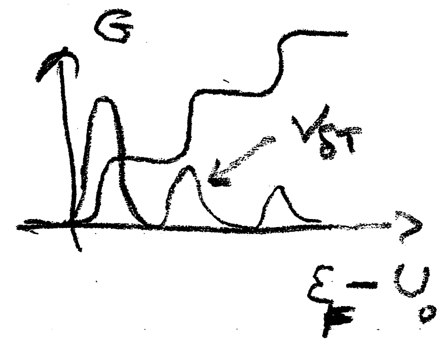
$\varphi_n(y)$: n-th eigenstate of $H = -\frac{\hbar^2}{2m}\partial_y^2 + \frac{b}{2}y^2$

$\psi_n(x)$ obey a 1D equation $(\varepsilon-\varepsilon_n)\psi = -\frac{\hbar^2}{2m}\psi'' - \frac{a}{2}x^2\psi + U_0\psi$

Transmission $T_n(\varepsilon) = \dfrac{1}{1+\exp\left(-\frac{2\pi}{\hbar\omega}(\varepsilon-\varepsilon_n-U_0)\right)}$ $\quad \omega = \sqrt{a/m}$

Conductance $G = \sum_n \frac{e^2}{h} T_n(\varepsilon_F)$

$$G = \sum_n \frac{e^2}{h}\frac{1}{e^{\frac{2\pi}{\hbar\omega}(U_0+\varepsilon_n-\varepsilon_F)}+1}$$



(ii) (5 pts) $I = \frac{e^2}{h}\int d\varepsilon \left(\sum_n T_n(\varepsilon)\right)\left(f_L(\varepsilon) - f_R(\varepsilon)\right)$

For $\mu_L = \mu_R$, $T_L \neq T_R$

$$I \approx \frac{e^2}{h} A \int d\varepsilon\,(\varepsilon-\varepsilon_F)\left(\frac{1}{e^{\beta_L(\varepsilon-\varepsilon_F)}+1} - \frac{1}{e^{\beta_R(\varepsilon-\varepsilon_F)}+1}\right)$$

$$A = \frac{\partial}{\partial\varepsilon}\left(\sum_n T_n(\varepsilon)\right)\Big|_{\varepsilon=\varepsilon_F}$$

From $\int_0^\infty x\,dx \sum_{m=1}^\infty \left(e^{-m\beta_L x} - e^{-m\beta_R x}\right)(-)^{m-1} =$

$$= \sum_{m=1}^\infty \frac{(-)^{m-1}}{m^2}\left(\beta_L^{-2} - \beta_R^{-2}\right) = \frac{\pi^2}{6}\left(1-\frac{1}{2}\right)\left((kT_L)^2 - (kT_R)^2\right)$$

$$I = \frac{\pi^2}{6}\left(-\frac{\partial G}{\partial U_0}\right)\left((kT_L)^2 - (kT_R)^2\right)$$

For $\delta T = T_L - T_R \ll T_{L,R}$, $I = \frac{\pi^2}{3}\left(-\frac{\partial G}{\partial U_0}\right) k_B^2 T\,\delta T$

Thermopower $V_{\delta T} = \frac{I_{\delta T}}{G} = \frac{\pi^2}{3}\,-\frac{\partial}{\partial U_0}(\log G)\,k_B^2 T\,\delta T$

Problem 2 (i) $S(\varepsilon) = \sum_\alpha e^{2i\theta_\alpha(\varepsilon)} |\alpha(\varepsilon)\rangle\langle\alpha(\varepsilon)|$

(4 pts) Because $S(\varepsilon)$ is unitary for real ε we have $\theta_\alpha(\varepsilon^*) = (\theta_\alpha(\varepsilon))^*$ (from analyticity in ε)

Thus $\left|e^{2i\theta_\alpha(\varepsilon^*)}\right| = \left|e^{2i\theta_\alpha(\varepsilon)}\right|^{-1}$

which indicates that there's a pole at $\varepsilon = z$ whenever there's a zero at $\varepsilon = z$

(ii) From the Fabry-Perot model

(6 pts) $t_{tot} = \frac{t^2}{1 - r^2 e^{2ika}}$ $\quad r_{tot} = r\,\frac{-2i\sin ka}{1-r^2 e^{2ika}}$

where $t = \frac{1}{1 - \lambda/2ik} \approx -\frac{2ik}{\lambda}$, $r \approx -\sqrt{1-\left(\frac{2k}{\lambda}\right)^2}$ ← ignore phase

obtained for a δ-function at large λ

Resonances: $e^{2ika} = r^{-2}$

Solving $2ika \approx \left(\frac{2k}{\lambda}\right)^2 + 2\pi i n \Rightarrow k_n a = \pi n - \frac{2(\pi n)^2}{a^2\lambda^2} i$

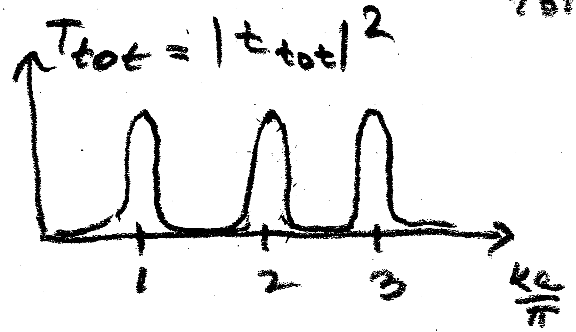
$$k_n = \frac{\pi n}{a} - \frac{2(\pi n)^2}{\lambda^2 a^3} i \equiv k_n' + i k_n''$$

Near nth resonance

$$S = e^{i\alpha_{reg}} \frac{k - k_n^*}{k - k_n} |-\rangle\langle-| + e^{i\beta_{reg}} |+\rangle\langle+| \quad (n \text{ odd})$$

$$S = e^{i\alpha_{reg}} \frac{k - k_n^*}{k - k_n} |+\rangle\langle+| + e^{i\beta_{reg}} |-\rangle\langle-| \quad (n \text{ even})$$

Transmission $t_{tot}(k = k_n) = 1$, $r_{tot}(k = k_n) = 0$



Problem 3.

(i) (2 points)

$$\epsilon\psi = -\frac{1}{2}\psi'' - \frac{1}{2}x^2\psi$$

$\psi_0(x) = e^{ix^2/2}$ $\quad \psi_0''(x) = (i - x^2)\psi_0(x) \Rightarrow$

$\psi_0(x)$ is eigenstate with energy $\epsilon_0 = -\frac{i}{2}$.

At large positive $x \approx X + \delta x$ $\quad \psi_0(x) \approx e^{i\frac{X^2}{2}} e^{iX\delta x + O(\delta x^2)}$,

which indicates that only outgoing wave is present.

Similarly, at large negative X, $\psi_0(x) \propto e^{-i|X|\delta x}$,

and again there is only outgoing wave.

$$a = 2^{-1/2}\left(x + i\frac{d}{dx}\right), \quad a^+ = 2^{-1/2}\left(x - i\frac{d}{dx}\right)$$

$$[a, a^+] = i$$

$$H = -a^+a - i/2 \Rightarrow$$

(move a through $(a^+)^n$, use $[a,a^+] = i$ n times)

$$H\left(a^{+n}|\psi_0\rangle\right) = -i\left(n+\frac{1}{2}\right)\left(a^{+n}|\psi_0\rangle\right)$$

n - even corresponds to even wavefunction,

n-odd — to odd WF.

(ii) (4 points) Scattering potential is symmetric $\Rightarrow$

scatt. matrix is diagonal in the even and odd basis,

$$S = e^{2i\theta_+}|+\rangle\langle+| + e^{2i\theta_-}|-\rangle\langle-|$$

In the $|1\rangle, |2\rangle$ basis thus gives

$$S = \begin{bmatrix} \frac{1}{2}(e^{2i\theta_+} + e^{2i\theta_-}) & \frac{1}{2}(e^{2i\theta_+} - e^{2i\theta_-}) \\ \frac{1}{2}(e^{2i\theta_+} - e^{2i\theta_-}) & \frac{1}{2}(e^{2i\theta_+} + e^{2i\theta_-}) \end{bmatrix}$$

Therefore, we have

$$f(\epsilon) = \frac{t(\epsilon)}{r(\epsilon)} = \frac{e^{2i\theta_+} - e^{2i\theta_-}}{e^{2i\theta_+} + e^{2i\theta_-}}$$

$\boxed{\text{4 points}}$

(iii) $e^{2i\theta_+(\varepsilon)}$ has poles when $\varepsilon = \mathcal{E}_n$, $n \to$ even

(because θ_+ is the phase shift for even channel)

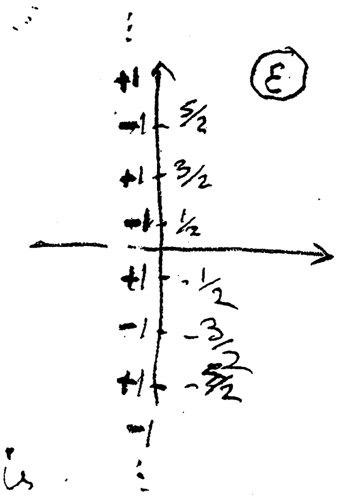
likewise, $e^{2i\theta_-(\varepsilon)}$ has poles when $\varepsilon = \mathcal{E}_n$, n odd.

It follows from Problem 1 that

$e^{2i\theta_+(\varepsilon)}\left(e^{2i\theta_-(\varepsilon)}\right)$ has zeroes at $\varepsilon = -\mathcal{E}_n$, n even (odd).

Therefore,

$$f(\varepsilon) = \begin{cases} +1, & \varepsilon = \mathcal{E}_n, \ n \text{ even} \\ -1, & \varepsilon = -\mathcal{E}_n, \ n \text{ even} \\ +1, & \varepsilon = -\mathcal{E}_n, \ n \text{ odd} \\ -1, & \varepsilon = \mathcal{E}_n \ \ n \text{ odd} \end{cases}$$



An analytic function which has such values is

$$f(\varepsilon) = i e^{\pi \varepsilon}$$

From $f(\varepsilon) = \frac{t(\varepsilon)}{r(\varepsilon)} = i\, e^{\pi \varepsilon}$ obtain:

$$|t|^2 = |r|^2 e^{2\pi\varepsilon}$$

combined w. $|t|^2 + |r|^2 = 1$

$$\boxed{|r|^2 = \frac{1}{1+e^{2\pi\varepsilon}} \qquad |t|^2 = \frac{e^{2\pi\varepsilon}}{1+e^{2\pi\varepsilon}}}$$