# 8.513 HW #1

## 1 Bound states in 1D

a) $-\frac{\hbar^2}{2m}\psi'' + \alpha\,\delta(x)\,\psi = E\psi$

(5pt) Matching values of $\psi(x \gtrless 0)$ and $\psi'(x \gtrless 0)$

find $\psi(0+) = \psi(0-)$

$$\psi'(0+) - \psi'(0-) = \frac{2\alpha m}{\hbar^2}\psi(0)$$

For

$\psi(x>0) = \psi(0)\,e^{-kx}$, $\psi(x<0) = e^{kx}\psi(0)$

have $-2k = \frac{2\alpha m}{\hbar^2} \longrightarrow \boxed{k = -\frac{\alpha m}{\hbar^2}}$

$E = -\frac{\alpha^2 m}{2\hbar^2}$ Need $\alpha < 0$

b) $E\psi_n = t(\psi_{n-1} + \psi_{n+1}) + V_n\psi_n$

(5pt) $V_n = \begin{cases} 0 & n \neq 0 \\ \varepsilon_0 & n = 0 \end{cases}$ For $\psi_n \propto e^{ikn}$, $E = 2t\cos k$

Take $\psi_n = \psi_0\,e^{-k|n|}$ as in part a)

For $n \neq 0$ S.E. gives $E = 2t\cosh k$

For $n = 0$, have $E - \varepsilon_0 = 2t\,e^{-k}$

solve for k: $\varepsilon_0 = 2t\sinh k$

Find $E = 2t\cosh\left(\sinh^{-1}\left(\varepsilon_0/2t\right)\right)$

Two cases:

$\varepsilon_0 > 0$ $E = \sqrt{\varepsilon_0^2 + (2t)^2}$

$\varepsilon_0 < 0$ $E = -\sqrt{\varepsilon_0^2 + (2t)^2}$ (b/c $\mathrm{Im}\,\sinh\frac{\varepsilon_0}{2t} = \pi$)

2. Localized & extended states

a) $\frac{d}{dx} W = \frac{d}{dx}(\psi_1 \psi_2' - \psi_2 \psi_1') = \psi_1 \psi_2'' - \psi_2 \psi_1'' = 0$

(5pt)

Thus $W(\psi_1, \psi_2) = $ const

But for $\psi_1 \sim e^{-k|x|}$ at large $x$

and $\psi_2 \sim \sum_j a_j e^{ik_j x}$

$W(\psi_1, \psi_2)$ is not a constant

Thus a localized and extended state cannot occur at the same energy

b)

(5pt)

3 S-matrix

a) A scattering state incident from the left side is

(5pt)

$$\psi_n(k) = \begin{cases} e^{ikn} + b e^{-ikn} & n \le 0 \\ a e^{ikn} & n \ge 0 \end{cases}$$

where $a$ is transmission, $b$ is reflection amplitudes

Matching values at $n=0$

$1+b=a$, $(\varepsilon-\varepsilon_0)a = t(ae^{ik}+e^{-ik}+be^{ik})$

Solving for $a, b$ find

$$a = \frac{i\sin k}{\frac{\varepsilon_0}{2t}+i\sin k}, \quad b = -\frac{\varepsilon_0/2t}{\frac{\varepsilon_0}{2t}+i\sin k}$$

$$v = \frac{\varepsilon-\varepsilon_0}{t}$$

$$S = \begin{pmatrix} b & a \\ a & b \end{pmatrix}$$

b) $$|a|^2+|b|^2 = \frac{(2\sin k)^2+(v-2\cos k)^2}{|2e^{ik}-v|^2} = 1$$

$$ba^*+ab^* = 0 \quad \rightarrow \quad \hat{S} \text{ unitary}$$

c) $$T = \frac{\sin^2 k}{1+(v/2)^2 - v\cos k} \qquad R = 1-T$$

(5pt)

$$e^{i\theta_1}e^{i\theta_2} = \det S = \frac{(v-2\cos k)^2+4\sin^2 k}{(v-2e^{ik})^2} = \frac{\frac{\varepsilon_0}{2t}-i\sin k}{\frac{\varepsilon_0}{2t}+i\sin k}$$

$$e^{i\theta_1}+e^{i\theta_2} = \operatorname{tr} S = 2b = -1-\frac{v-2e^{-ik}}{v-2e^{ik}}$$

Thus $$e^{i\theta_1} = \frac{\frac{\varepsilon_0}{2t}-i\sin k}{\frac{\varepsilon_0}{2t}+i\sin k}, \quad e^{i\theta_2} = -1$$

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