1. Weak localization

Consider the weak localization contribution to conductivity, expressed through the Cooperon $\left(D(-i\nabla - 2e\mathbf{A}(r))^2 - i\omega + \frac{1}{\tau_\phi}\right) C_\omega(r, r') = \delta(r - r')$ as follows:

$$\delta\sigma_{WL} = -\frac{4De^2}{h} \lim_{\omega\to 0} C_\omega(r = r') = -\frac{4De^2}{h} \int \frac{d^d q}{(2\pi)^d} \frac{1}{Dq^2 + \frac{1}{\tau_\phi}}$$

where the last expression is valid in the absence of magnetic field, $\mathbf{A}(r) = 0$.

a) Find the dependence of $\delta\sigma_{WL}$ on τ_ϕ in space dimension $d = 2, 3, 1$.

b) Suppose that electron temperature is very small, so that the dephasing length $L_\phi = \sqrt{D\tau_\phi}$ exceeds the system size L. In this case transport is fully coherent, and thus we can approximate $\tau_\phi \approx \infty$. Find the dependence of weak localization conductivity $\delta\sigma_{WL}$ on system size L for $D = 2, 3, 1$.

2. Suppression of weak localization by magnetic field

a) Show that in the presence of a uniform magnetic field, the weak localization contribution to conductivity can be brought to the form

$$\delta\sigma_{WL}(B) = -\frac{4De^3 B}{\hbar c} \int \frac{dq_z}{2\pi} \sum_{n\geq 0} \frac{1}{Dq_z^2 + \Omega_*(n + \frac{1}{2}) + \frac{1}{\tau_\phi}}$$

where $\Omega_* = 4eDB/\hbar c$ (in three dimensions).

b) In the regime of strong magnetic field $\Omega_*\tau_\phi \gg 1$ (or, weak dephasing), find the dependence $\delta\sigma_{WL}(B)$ *vs.* B. Show that magnetoresistance is *negative*: magnetic field suppresses weak localization and thereby *enhances* conductivity.

3. Aharonov-Bohm effect.

a) Consider a thin metallic cylinder in a magnetic field parallel to the cylinder axis, which induces flux $\Phi = \pi R^2 B$ through the cylinder cross-section. Show that the weak localization conductivity for such a system is given by

$$\delta\sigma_{WL}(\Phi) = -\frac{2e^2}{\pi\hbar R} \sum_{m=-\infty}^{\infty} \int \left[q_z^2 + \frac{1}{R^2}\left(m - \frac{2\Phi}{\Phi_0}\right)^2 + \frac{1}{D\tau_\phi}\right]^{-1} \frac{dq_z}{2\pi}$$

where $\Phi_0 = hc/e$ is a single-electron flux quantum. Thus the period of oscillations in the dependence $\delta\sigma_{WL}(B)$ *vs.* B equals the superconducting flux quantum $\Phi_0/2 = \hbar c/2e$.

b) Show that

$$\delta\sigma_{WL}(\Phi) = -\frac{2e^2}{\pi\hbar} \sum_{k=-\infty}^{\infty} e^{4\pi i\Phi/\Phi_0} K_0(2\pi kR/L_\phi)$$

where $L_\phi = \sqrt{D\tau_\phi}$ is the dephasing length and K_0 is the Macdonald function. Analyze $\delta\sigma_{WL}$ for $R \gg L_\phi$ with the help of the asymptotic form $K_0(x \gg 1) \approx \sqrt{\frac{\pi}{2x}}e^{-x}$.

Hints: Use the Poisson summation formula

$$\sum_{m=-\infty}^{\infty} f(m) = \sum_{k=-\infty}^{\infty} \tilde{f}(k), \quad \tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{2\pi i k x} dx$$

and the integral identity for the Macdonald function

$$\int_0^{\infty} \frac{\cos tx \, dt}{(t^2+1)^{\nu}} = \left(\frac{x}{2}\right)^{\nu} \frac{\Gamma(1/2)}{\Gamma(\nu+1/2)} K_{\nu}(x).$$

See:

`http://en.wikipedia.org/wiki/Bessel_function`
`and`
`http://mathworld.wolfram.com/MacdonaldFunction.html`