1. Two-particle propagator in the $\langle GG\rangle = \langle G\rangle\langle G\rangle$ approximation.

Consider a density-density correlator in the Fermi gas with disorder, $\langle n_{q,\omega} n_{q,\omega}\rangle$, where $n_{q,\omega} = \int\int e^{i\omega t - iqr}\psi^\dagger(r,t)\psi(r,t)d^3r dt$ are Fourier harmonics of particle density. Assuming the time and position dependence to be slow, $\hbar\omega \ll E_F$, $\hbar q \ll p_F$, we can evaluate this correlator using Matsubara Greens functions linearized near Fermi level, $G(i\epsilon, \xi) = \frac{1}{i\epsilon - \xi}$.

a) Starting with the general expression for the correlator, $\int \frac{d\epsilon}{2\pi}\,\mathrm{Tr}\,(G(i\epsilon + i\omega, \mathbf{p}_+)G(i\epsilon, \mathbf{p}_-))$, $(p_\pm = p \pm q/2)$ and averaging it over gaussian disorder, show that in the $\langle GG\rangle = \langle G\rangle\langle GG\rangle$ approximation it becomes

$$B(\omega, q) = \frac{\nu}{2}\int \frac{do_{\mathbf{p}}}{|\omega| + i\mathbf{qv}\,\mathrm{sgn}\,\omega + \frac{1}{\tau}}$$

Evaluate this expression, which represents a single rung of a diffusion ladder diagram, at $\omega\tau \ll 1$ and $q\ell \ll 1$.

b) Continue the expression for $B(\omega, q)$ from positive imaginary axis $i\omega$ to the real values of ω. By taking Fourier transform in ω and $\mathbf{q}$ show that in position representation $B(t, \mathbf{r}) = \frac{\nu}{2r^2}\delta(r - v_F t)e^{-r/\ell}$, where $r = |\mathbf{r}|$. Interpret this result quasiclassically.

2. AC conductivity of an ideal Fermi gas.

The Kubo formula can be used to express response functions of a many-body system in terms of correlators involving Matsubara Greens functions $G(i\epsilon, \xi) = \frac{1}{i\epsilon - \xi}$. Consider an ideal Fermi gas (without disorder!) perturbed by an external electric field $\mathbf{E}(\mathbf{r}, t) \propto \mathbf{E}e^{-i\omega t + i\mathbf{qr}} + \mathrm{c.c.}$. Assume the time and position dependence to be slow, $\hbar\omega \ll E_F$, $\hbar q \ll p_F$.

Write an expression for linear response of electric current $j_\alpha = \sigma_{\alpha\beta}E_\beta$ in terms of Greens functions, and find the AC conductivity $\sigma_{\alpha\beta}(\omega, q)$. Consider two cases: a) $\mathbf{E} \perp \mathbf{q}$; b) $\mathbf{E} \parallel \mathbf{q}$.

Show that in both cases the results are identical to those found from collisionless Boltzmann equation.

3. Greens function for a random walk

Greens functions can sometimes be useful even in the problems that have nothing to do with quantum mechanics. Here we show how Greens function can be applied to study statistics of random walks.

Consider a random walk on the n-dimensional cubic lattice, which starts at $t = 0$ at the origin, $\mathbf{x} = 0$. At times $t > 0$, at each step, the walker can move with equal probability from current to any of $2n$ neighboring sites. We denote by $p(t, \mathbf{x})$ the probability to find the walker on a site $\mathbf{x} = (x_1, ...x_n)$ at time t. (In this problem time and space are discrete.) In probability theory it is convenient to describe the process of random walk by so-called *generating function*

$$G(z, q) = \sum_{t,\mathbf{q}} z^t e^{\mathbf{qx}} p(t, \mathbf{x}), \quad (t \geq 0,\ |z| \leq 1)$$

which is in many ways analogous to the Greens function derived in Quantum Mechanics for the single particle Schrödinger equation.

a) Show that

$$G(z, q) = \frac{1}{1 - zW(\mathbf{q})}, \quad W(\mathbf{q}) = \frac{1}{n}(\cos q_1 + ... \cos q_n)$$

b) Argue that the behavior of $p(t, \mathbf{x})$ at long times $t \gg 1$ can be obtained by linearizing $G(z, q)$ near $\mathbf{q} = 0$ and $z = 1$. Replace z by $e^{i\omega}$, linearize in $\omega \ll 1$ to obtain the Greens function of the diffusion equation, and use it to find $p(t, \mathbf{x})$ at $t \gg 1$

c) [a hard one] There are many interesting questions about random walks that can be addressed with the help of this Greens function. Here we will be concerned with the probability of returns to $\mathbf{x} = 0$ at long times. Since at $t \gg 1$ the probability distribution found in b) behaves as $t^{-n/2}$, the integral $\int_0^\infty p(t, 0)dt$ diverges at large t for $n = 1, 2$ and converges at $n > 2$. Thus we anticipate that for $n = 1, 2$ the random walk passes infinitely many times through every lattice site, whereas for $n > 2$ the number of passages through each site is at most finite.

Consider the probability P that the walk never returns to the origin at $t > 0$. Show that this quantity can be expressed through the generating function $G(z, q)$ as follows:

$$P^{-1} = \int G(1, q) \frac{d^n q}{(2\pi)^n}$$

where the integral over $\mathbf{q}$ is taken over the period in the reciprocal space $-\pi < q_i < \pi$ (Brillouin zone). Check that this expression indeed gives $P = 0$ at $n = 1, 2$ and $0 < P < 1$ at $n > 2$.

Hint: Consider the generating function $\tilde{G}(z, \mathbf{q})$ of random walks that start at $\mathbf{x} = 0$ at $t = 0$ but never come back to $\mathbf{x} = 0$ at later times. The quantity $\tilde{G}$ is analogous to the Greens function of a QM particle moving in the field of a repulsive potential. In particular, it can be related with a suitable "T-matrix" by Dyson's equation, which can be used to express $\tilde{G}$ through G.