Greens functions and Kubo formula

1. The density of states.

a) Show that the single-particle Greens function

$$G^R(\epsilon) = \sum_p \frac{|p\rangle\langle p|}{\epsilon - p^2/2m + i\delta}, \tag{1}$$

where $|p\rangle = \frac{e^{i\mathbf{px}/\hbar}}{\sqrt{V}}$, $\sum_p ... \equiv V\int \frac{d^dp}{(2\pi\hbar)^d}...$, can be used to find the density of states as follows

$$\nu(\epsilon) \equiv dN/d\epsilon = -\frac{1}{\pi}\mathrm{Im}\,\mathrm{Tr}\,G^R(\epsilon)$$

b) Use this relation to compute the density of states for free particle in space dimension $d = 1, 2, 3$.

2. Kubo formula and Greens functions.

Consider a quantum mechanical system perturbed by an external field $f(t)$ that couples to the quantity $\widehat{B}$ as $\widehat{H}(t) = \widehat{H}_0 + \widehat{B}f(t)$.

a) Show that the linear response function of a quantity $\widehat{A}$ to a harmonically varying field $f(t) = f_\omega e^{-i\omega t} + \text{c.c.}$ is given by

$$\langle A\rangle_\omega = \chi(\omega)f_\omega, \quad \chi(\omega) = \frac{i}{\hbar}\int_0^\infty \langle [A(t), B(0)]\rangle e^{i\omega t}dt$$

where $\langle ...\rangle = \mathrm{Tr}\,(...\widehat{\rho})$ stands for averaging over the system state described by the density matrix $\widehat{\rho}$ and $\langle ...\rangle_\omega$ means the Fourier harmonic at frequency ω.

b) Consider quantities A, B of a bilinear many-body form, $\widehat{A}(t) = \sum_{mk} A_{mk}a_m^+ a_k e^{-i(E_k - E_m)t}$, $\widehat{B}(t) = \sum_{mk} B_{mk}a_m^+ a_k e^{-i(E_k - E_m)t}$, where a_m, a_m^+ are fermion creation and annihilation operators. Show that the response function $\chi(\omega)$ can be obtained from the Matsubara imaginary-frequency response function

$$\chi(i\omega_{n'}) = -T\sum_{\epsilon_n = \pi(2n+1)T} \mathrm{Tr}\left(G^M(i\epsilon_n + i\omega_{n'})\widehat{B}G^M(i\epsilon_n)\widehat{A}\right), \quad G^M(i\epsilon_n) = (i\epsilon_n - H_0)^{-1}$$

where $\omega_{n'}$ takes discrete values $2\pi n'T$. Show that $\chi(\omega)$ with real frequency ω is given by the analytic continuation from discrete frequencies on the positive imaginary axis $i\omega_{n'>0} \to \omega$.

Use the identity: $\sum_n \frac{1}{z - \pi T(2n+1)i} = \frac{\beta}{2}\tanh\frac{\beta z}{2}$, $\beta \equiv 1/T$.

3. Disorder-averaged Greens function in position-space representation.

a) Show that the Greens function of a free particle, Eq.(1), in position space (in three dimensions) is given by

$$G^R(\epsilon, \mathbf{x}, \mathbf{x}') \equiv \langle \mathbf{x}|\widehat{G}^R(\epsilon)|\mathbf{x}'\rangle = -\frac{m}{2\pi\hbar^2}\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}, \quad k = \sqrt{2m\epsilon}/\hbar \tag{2}$$

b) We found that the disorder-averaged Greens function takes the form

$$\langle G^R(\epsilon)\rangle = \sum_p \frac{|p\rangle\langle p|}{\epsilon - p^2/2m + i\gamma/2}, \quad \gamma = 2\pi\lambda\,\nu(\epsilon)$$

(ignoring the real part of self-energy; see Lectures 11,12). Show that in position space this Greens function can be written as

$$\langle G^R(\epsilon, \mathbf{x}, \mathbf{x}') \rangle = G_0^R(\epsilon, \mathbf{x}, \mathbf{x}') e^{-|\mathbf{x}-\mathbf{x}'|/2\ell}$$

where $G_0^R(\epsilon, \mathbf{x}, \mathbf{x}')$ is the Greens function in the absence of disorder, Eq.(2), and $\ell = v/\gamma = p/m\gamma$ is the mean free scattering path. Assume that scattering on disorder is weak: $\gamma \ll \epsilon/\hbar$, or $\lambda \ll \ell$.