1. Boundary scattering.

Consider degenerate two-dimensional electrons moving in a strip of width W. The system is described by the Boltzmann equation

$$(\partial_t + \mathbf{v}\cdot\nabla_r + e\mathbf{E}\cdot\nabla_p)f(t,r,p) = \frac{1}{2\pi\tau}\int d\theta_{p'}(f(t,r,p') - f(t,r,p)), \quad -\frac{1}{2}W < x < \frac{1}{2}W,$$

where τ is the mean free scattering time describing collisions with short-range disorder.

a) The boundary conditions at the strip edges $x = \pm\frac{1}{2}W$ can be set up to describe specular scattering as follows

$$x = \pm\frac{1}{2}W:\ f(-\theta_p) = f(\theta_p),$$

where θ_p is the angle between electron velocity and the strip boundary. Consider electric current driven along the strip, $\mathbf{E} \parallel \hat{y}$. Show that the resistance of the strip per unit length is $m/ne^2\tau W$, i.e. it is unaffected by the presence of the boundary.

b) In the case when scattering on the boundary is of diffuse character, such that electron velocity is fully randomized after each collision, the boundary condition for the Boltzmann equation can be written as follows:

$$x = \frac{1}{2}W:\ f(-\pi < \theta_p < 0) = \frac{1}{2}\int_0^\pi d\theta' f(\theta')\sin\theta';$$

$$x = -\frac{1}{2}W:\ f(0 < \theta_p < \pi) = \frac{1}{2}\int_{-\pi}^0 d\theta' f(\theta')|\sin\theta'|$$

Find the solution of B.k.e. describing current driven by a weak electric field, and use it to determine the resistance. Consider the limits $W \ll \ell = v_F\tau$ and $W \gg \ell$.

See: H. Sondheimer, Adv. Phys. 1, 1 (1952) [you can download PDF from course webpage], and C. W. J. Beenakker and H. van Houten review cond-mat/0412664 “Quantum transport in semiconductor nanostructures.”

2. T-matrix for QM particle scattering[1]

Consider a particle of mass m moving freely and scattering on a potential $V(\mathbf{r})$, described by $H = p^2/2m + V(\mathbf{r})$. In this case, the T-matrix operator is defined in terms of the particle Greens function $G(\epsilon) = (\epsilon - H + i\delta)^{-1}$ as

$$\langle\mathbf{k}'|G(\epsilon)|\mathbf{k}\rangle = G_0(\epsilon,\mathbf{k})(2\pi)^d\delta^d(\mathbf{k}-\mathbf{k}') + G_0(\epsilon,\mathbf{k}')T(\epsilon,\mathbf{k}',\mathbf{k})G_0(\epsilon,\mathbf{k}),$$

where $G_0(\epsilon,\mathbf{k}) = (\epsilon - k^2/2m + i\delta)^{-1}$ and d is space dimension.

a) In $d = 3$, relate the T-matrix to the scattering amplitude, defined by the asymptotic form of scattering state,

$$\psi(\mathbf{r}) = e^{i\mathbf{kr}} + \frac{f(\mathbf{k}',\mathbf{k})}{|\mathbf{r}|}e^{ik'|\mathbf{r}|}, \quad (\mathbf{r}\text{ large}).$$

b) In $d = 1$, relate the T-matrix to the 2×2 S-matrix.

[1]Problems 2 and 3 are here in a way of reviewing some useful facts from one-particle QM. Some people may find the QM textbook by Sakurai (Chap.7) helpful.

3. Scattering on a delta function potential.

Consider a particle scattering on a potential in $d = 3$ at low kinetic energy such that deBroglie wavelength of the particle is long compared to the potential radius r_0.

a) Argue that for a potential of radius r_0 the T-matrix $T_{\mathbf{k}',\mathbf{k}}$ varies with $\mathbf{k}$, $\mathbf{k}'$ on the wavenumber scale r_0^{-1}. Show that at low energy scattering becomes purely s-wave, i.e. the scattering amplitude is isotropic and angle-independent: $f(\mathbf{k}', \mathbf{k}) = a$. (The quantity a is called the scattering length.)

b) By using an equation for the T-matrix or otherwise, find the energy dependence $T(\epsilon)$ at low energy $\epsilon \ll \hbar^2/2ma^2$, when scattering is predominantly s-wave. Show that $T(\epsilon) - T(0) \approx C\sqrt{-\epsilon}$, where the coefficient C can be itself expressed through $T(0)$. From that, find energy dependence of the scattering amplitude f at low energy.