Scattering theory of transport, III

1. Wigner's scattering time.

In this problem we consider scattering on a resonance in the time domain. For simplicity, we shall focus on the one-channel system discussed in Lecture 4: a delta-function potential $U(x) = \alpha\delta(x-a)$ on a half-line $x > 0$ with a hard-wall boundary condition at $x = 0$. Suppose that this system has a quasi-bound state with the complex energy $E = E_0 - \frac{i}{2}\gamma$.

(i) Consider an incident wavepacket with energy $\bar{E}$ close to E_0 and narrow dispersion $\Delta\epsilon \ll \gamma$ (this corresponds to large spatial width $\Delta x \gg \hbar v_F/\gamma$). Show that after scattering the wavepacket comes out with a time lag

$$\Delta\tau = 2\hbar(d\delta(E)/dE)_{E=\bar{E}} \tag{1}$$

where $\delta(E)$ is scattering phase defined by $e^{2i\delta} = -\frac{E-E_0-\frac{i}{2}\gamma}{E-E_0+\frac{i}{2}\gamma}$.

Analyze the time lag as a function of energy for this resonance model. What is the maximal value of $\Delta\tau$?

(ii) Now consider a wide wavepacket, $\Delta E \gg \gamma$, which corresponds to narrow spatial width $\Delta x = \hbar v/\Delta\epsilon \ll \hbar v_F/\Gamma$. Show that the reflected wavepacket consists of a non-delayed and delayed parts. Find the probability for the scattering particle to be delayed by a time τ. Interpret the result.

2. Multichannel transport

For an electron system having two external leads with N channels in each lead, left and right, show that the scattering matrix can be written as

$$S = \begin{pmatrix} r_{11} & t_{21} \\ t_{12} & r_{22} \end{pmatrix} \tag{2}$$

where r_{11}, r_{22}, t_{21}, t_{12} are $N \times N$ matrices describing reflection and transmission among N channels of each lead.

(i) Evaluate electric current between the leads as a difference of the left-to-right and right-to-left currents. Show that the total current is equal to

$$I = \frac{e}{\hbar}\int \frac{d\epsilon}{2\pi}\left[f_1(\epsilon)\,\mathrm{Tr}\,(t_{21}t_{21}^\dagger) - f_2(\epsilon)\,\mathrm{Tr}\,(t_{12}t_{12}^\dagger)\right] \tag{3}$$

where $f_{1,2}(\epsilon)$ are particle energy distributions in reservoirs. (Note that $\mathrm{Tr}AA^\dagger = \sum_{i,j=1\ldots N}|A_{ij}|^2$.)

(ii) Apply the result (3) to current between two reservoirs which supply equilibrium distributions, with external voltage controlling the chemical potential difference, $\mu_L = \mu_R + eV$. Use unitarity of S to show that $\mathrm{Tr}\,(t_{21}t_{21}^\dagger) = \mathrm{Tr}\,(t_{12}t_{12}^\dagger)$. Take the limit $V \to 0$ to obtain Ohm's law $I = GV$ with conductance given by the multi-channel Landauer formula

$$G = \frac{e^2}{h}\,\mathrm{Tr}\,(t_{21}t_{21}^\dagger) \tag{4}$$

What are the maximal and minimal possible values of G?

3. Drude-Lorentz conductivity of a Fermi gas

Consider electrons at density n in a 2D metal at zero temperature in the presence of randomly placed point-like scatterers, $U(r) = \sum_i u\delta(\mathbf{r} - r_i)$. The concentration of scatterers is n_{imp}.

a) Treating scattering in the Born approximation, estimate the transition rate $w(\mathbf{p}', \mathbf{p})$ in the collision integral of the Boltzmann equation. Find the mean free path and evaluate Drude conductivity.

b) In the presence of a finite magnetic field, find the components of the resistivity tensor $\rho_{xx}(B)$ and $\rho_{xy}(B)$.

See: C. W. J. Beenakker and H. van Houten review cond-mat/0412664 "Quantum transport in semiconductor nanostructures" pages 8-11.