1. Wigner’s scattering time.

In this problem we consider scattering on a resonance in the time domain. For simplicity, we shall focus on the one-channel system discussed in Lecture 4: a delta-function potential \( U(x) = \alpha \delta(x - a) \) on a half-line \( x > 0 \) with a hard-wall boundary condition at \( x = 0 \). Suppose that this system has a quasi-bound state with the complex energy \( E = E_0 - \frac{i}{2} \gamma \).

(i) Consider an incident wavepacket with energy \( \overline{E} \) close to \( E_0 \) and narrow dispersion \( \Delta \epsilon \ll \gamma \) (this corresponds to large spatial width \( \Delta x \gg \hbar v_F/\gamma \)). Show that after scattering the wavepacket comes out with a time lag \( \Delta \tau = 2\hbar (d\delta(E)/dE)_{E=E_0} \) (1)

where \( \delta(E) \) is scattering phase defined by \( e^{2i\delta} = -\frac{E-E_0-\frac{i}{2} \gamma}{E-E_0+\frac{i}{2} \gamma} \).

Analyze the time lag as a function of energy for this resonance model. What is the maximal value of \( \Delta \tau \)?

(ii) Now consider a wide wavepacket, \( \Delta E \gg \gamma \), which corresponds to narrow spatial width \( \Delta x = \hbar v/\Delta \epsilon \ll \hbar v_F/\Gamma \). Show that the reflected wavepacket consists of a non-delayed and delayed parts. Find the probability for the scattering particle to be delayed by a time \( \tau \). Interpret the result.

2. Multichannel transport

For an electron system having two external leads with \( N \) channels in each lead, left and right, show that the scattering matrix can be written as

\[
S = \begin{pmatrix} r_{11} & t_{21} \\ t_{12} & r_{22} \end{pmatrix}
\] (2)

where \( r_{11}, r_{22}, t_{21}, t_{12} \) are \( N \times N \) matrices describing reflection and transmission among \( N \) channels of each lead.

(i) Evaluate electric current between the leads as a difference of the left-to-right and right-to-left currents. Show that the total current is equal to

\[
I = \frac{e}{h} \int \frac{d\epsilon}{2\pi} \left[ f_1(\epsilon) \text{Tr} (t_{21} t_{21}^\dagger) - f_2(\epsilon) \text{Tr} (t_{12} t_{12}^\dagger) \right]
\] (3)

where \( f_{1,2}(\epsilon) \) are particle energy distributions in reservoirs. (Note that \( \text{Tr} A A^\dagger = \sum_{i,j=1,...,N} |A_{ij}|^2 \).)

(ii) Apply the result (3) to current between two reservoirs which supply equilibrium distributions, with external voltage controlling the chemical potential difference, \( \mu_L = \mu_R + eV \). Use unitarity of \( S \) to show that \( \text{Tr} (t_{21} t_{21}^\dagger) = \text{Tr} (t_{12} t_{12}^\dagger) \). Take the limit \( V \to 0 \) to obtain Ohm’s law \( I = GV \) with conductance given by the multi-channel Landauer formula

\[
G = \frac{e^2}{h} \text{Tr} (t_{21} t_{21}^\dagger)
\] (4)

What are the maximal and minimal possible values of \( G \)?
3. Drude-Lorentz conductivity of a Fermi gas

Consider electrons at density $n$ in a 2D metal at zero temperature in the presence of randomly placed point-like scatterers, $U(r) = \sum_i u \delta(r - r_i)$. The concentration of scatterers is $n_{imp}$.

a) Treating scattering in the Born approximation, estimate the transition rate $w(p', p)$ in the collision integral of the Boltzmann equation. Find the mean free path and evaluate Drude conductivity.

b) In the presence of a finite magnetic field, find the components of the resistivity tensor $\rho_{xx}(B)$ and $\rho_{xy}(B)$.

See: C. W. J. Beenakker and H. van Houten review cond-mat/0412664 “Quantum transport in semiconductor nanostructures” pages 8-11.