Scattering theory of transport, II

1. Quantum point contact.

One can model quantum point contact (QPC) in a two-dimensional electron gas [see van Wees, et al, Phys. Rev. Lett. 60, 848 (1988)] by a quadratic saddle-like potential:

\[ U(x, y) = -\frac{1}{2}ax^2 + \frac{1}{2}by^2 + U_0 \]  

(1)

The \( x \) dependence describes potential along conduction direction, the \( y \) dependence describes confinement in transverse direction. The parameters \( a \) and \( b \) depend on the geometry of the split gate used to create QPC; the energy \( U_0 \) is proportional to gate voltage \( V_g \) and can be varied.

(i) Identify scattering channels by separating variables in the 2d Schrödinger equation and discuss the two-terminal conductance \( G \) dependence on \( U_0 \) treating electrons as an ideal degenerate Fermi gas. Use the result for transmission coefficient, \( T(\epsilon) = |t|^2 = \frac{e^2}{\pi \hbar \omega} \left( \frac{e^2}{\pi \hbar \omega} + 1 \right) \), derived for 1d inverted parabolic potential \( U(x) = -\frac{1}{2}m\omega^2x^2 \).

(ii) Find the thermopower coefficient of the QPC. Consider the situation when a small temperature difference \( \delta T = T_L - T_R \) exists between the reservoirs, whereas no voltage is applied, \( V = \mu_L - \mu_R = 0 \). Find the electric current induced by temperature difference, \( I = A\delta T \), and determine the thermoelectric coefficients (see Lecture 2, page 7).

2. Breit-Wigner resonances.

Consider a general scattering matrix \( S \). Being unitary, \( S \) can be brought to a diagonal form:

\[ S = \sum_{\alpha=1...N} e^{2i\theta_\alpha}|\alpha\rangle\langle\alpha|, \]  

(2)

where \(|\alpha\rangle\) and \(\langle\alpha|\) are in- and out-states, and \(N\) is the number of scattering channels.

As a function of energy, the scattering matrix \( S(\epsilon) \) is analytic in the upper half-plane of complex \( \epsilon \). This condition expresses causality requirement. (Think of the relation \( \psi_{\text{out}} = S\psi_{\text{in}} \) from the cause-effect point of view and compare to Kramers-Kronig properties of susceptibility.)

Since \( S(\epsilon) \) is analytic in the upper complex half-plane, \( \text{Im} \epsilon > 0 \), the phase factors in Eq.(2) may have zeros in this half-plane, but no poles or other singularities (such as branch cuts, etc.). Show that each zero \( \epsilon = z_j \) must be accompanied by a pole at \( \epsilon = z_j^* \) in the lower half-plane \( \text{Im} \epsilon < 0 \).

(i) Suppose now that one of the zeros \( z_j = \epsilon_0 + \frac{i}{2}\Gamma \) is much closer to the real axis than other zeros. For \( \epsilon \) values close enough to \( \epsilon_0 \) the \( S \)-matrix can be approximated by Breit-Wigner (BW) model

\[ S = e^{2i\theta_n^{(\text{reg})}} \frac{\epsilon - \epsilon_0 - \frac{i}{2}\Gamma}{\epsilon - \epsilon_0 + \frac{i}{2}\Gamma} |n\rangle\langle n| + S^{(\text{reg})} \]  

(3)

where \( S^{(\text{reg})} \) and \( e^{2i\theta_n^{(\text{reg})}} \) describe the “regular” parts of the \( S \)-matrix which have no singularity at \( \epsilon = z_j^* \).

(ii) Transport through a quantum dot can be schematically described by a 1D Schroedinger equation with potential that confines particle between two barriers. As a simple model, we consider \( U(x) = \lambda\delta(x - a/2) + \lambda\delta(x + a/2) \). In this potential, the resonances \( \epsilon_{1,2,3,...} \) are described by the
standing wave condition \( k_n a \approx \pi n \). Consider the limit of strong barriers (large \( \lambda \)) and find the Breit-Wigner form of the S-matrix for near a resonance \( \epsilon_n \).

Find the transmission and reflection coefficients, \( T(\epsilon) = |t|^2 \) and \( R(\epsilon) = |r|^2 \), and analyze their energy dependence near a resonance.

### 3. Quasi-bound states and scattering for inverted parabola. (This is a hard one)

In this problem we analyze scattering on an inverted parabola, used in Problem 1 to model quantum point contact. Consider a 1d Schrödinger equation

\[
\epsilon \psi = -\frac{1}{2} \psi'' - \frac{1}{2} x^2 \psi
\]

We learned in Problem 2 that complex poles and zeros are related to quasi-bound states which have only the “out” but no “in” component. Here we demonstrate how one can reconstruct the S-matrix by analyzing these states.

(i) Show that the state \( \psi_0(x) = e^{ix^2/2} \) represents a quasi-bound state for complex energy \( \epsilon_0 = -i/2 \) which has only outgoing but no incoming component at infinity.

Write the Hamiltonian in the form \( H = -a^\dagger a - i/2 \), where

\[
a = 2^{-1/2}(x + id/dx), \quad a^\dagger = 2^{-1/2}(x - id/dx)
\]

are analogs of the ladder (lowering and raising) operators for this problem. From the commutators of these operators and \( H \), treating \( a, a^\dagger \) by analogy with harmonic oscillator problem, show that there are quasi-energy states with energies

\[
\epsilon_n = -i \left( n + \frac{1}{2} \right), \quad n \geq 0,
\]

where even and odd values of \( n \) correspond to even and odd states \( \psi_n(x) \).

(ii) Show that the 2 \( \times \) 2 scattering matrix for this problem can be diagonalized in the channel space in the even and odd basis:

\[
S = e^{2i\theta_+} |+\rangle \langle +| + e^{2i\theta_-} |-\rangle \langle -|, \quad \text{where} \quad |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},
\]

and \( \theta_\pm \) are energy-dependent scattering phases for the even and odd channels. From this representation, find the ratio of the transmission and reflection amplitudes,

\[
f(\epsilon) = t(\epsilon)/r(\epsilon)
\]

in terms of the phase factors \( e^{2i\theta_\pm} \).

(iii) Use the information obtained in part (i), combined with the result of Problem 1, to show that the quantities \( e^{2i\theta_+} \) (\( e^{2i\theta_-} \)) as a function of complex energy, have zeros at \( \epsilon = -\epsilon_n \) and poles at \( \epsilon = \epsilon_n \), where \( n \geq 0 \) is an even (odd) integer. Use this result to evaluate the function \( f(\epsilon) \) at the points \( \epsilon = \pm i(n + 1/2) \) of the complex \( \epsilon \) plane.

Having this information, reconstruct the function \( f(\epsilon) \) and to find the transmission and reflection coefficients for inverted parabola \( U(x) = -\frac{1}{2}m\omega^2x^2 \).

**Hint:** Two analytic functions of complex variable which coincide on a set of points \( z_n \) with a limiting point somewhere in the complex plane or at infinity, coincide in the entire plane.