## Scattering theory of transport, II

### 1. Quantum point contact.

One can model quantum point contact (QPC) in a two-dimensional electron gas [see van Wees, et al, Phys. Rev. Lett. 60, 848 (1988)] by a quadratic saddle-like potential:

$$U(x,y) = -\frac{1}{2}ax^2 + \frac{1}{2}by^2 + U_0 \tag{1}$$

The $x$ dependence describes potential along conduction direction, the $y$ dependence describes confinement in transverse direction. The parameters $a$ and $b$ depend on the geometry of the split gate used to create QPC; the energy $U_0$ is proportional to gate voltage $V_g$ and can be varied.

(i) Identify scattering channels by separating variables in the 2d Schrödinger equation and discuss the two-terminal conductance $G$ dependence on $U_0$ treating electrons as an ideal degenerate Fermi gas. Use the result for transmission coefficient, $T(\epsilon) = |t|^2 = e^{2\pi\epsilon/\hbar\omega}/(e^{2\pi\epsilon/\hbar\omega}+1)$, derived for 1d inverted parabolic potential $U(x) = -\frac{1}{2}m\omega^2x^2$.

(ii) Find the thermopower coefficient of the QPC. Consider the situation when a small temperature difference $\delta T = T_L - T_R$ exists between the reservoirs, whereas no voltage is applied, $V = \mu_L - \mu_R = 0$. Find the electric current induced by temperature difference, $I = A\delta T$, and determine the thermoelectric coefficients (see Lecture 2, page 7).

### 2. Breit-Wigner resonances.

Consider a general scattering matrix $S$. Being unitary, $S$ can be brought to a diagonal form:

$$S = \sum_{\alpha=1\ldots N} e^{2i\theta_\alpha}|\alpha\rangle\langle\alpha|, \tag{2}$$

where $|\alpha\rangle$ and $\langle\alpha|$ are in- and out-states, and $N$ is the number of scattering channels.

As a function of energy, the scattering matrix $S(\epsilon)$ is analytic in the upper half-plane of complex $\epsilon$. This condition expresses causality requirement. (Think of the relation $\psi_{out} = S\psi_{in}$ from the cause-effect point of view and compare to Kramers-Kronig properties of susceptibility.)

Since $S(\epsilon)$ is analytic in the upper complex half-plane, $\mathrm{Im}\,\epsilon > 0$, the phase factors in Eq.(2) may have zeros in this half-plane, but no poles or other singularities (such as branch cuts, etc.). Show that each zero $\epsilon = z_j$ must be accompanied by a pole at $\epsilon = z_j^*$ in the lower half-plane $\mathrm{Im}\,\epsilon < 0$.

(i) Suppose now that one of the zeros $z_j = \epsilon_0 + \frac{i}{2}\Gamma$ is much closer to the real axis than other zeros. For $\epsilon$ values close enough to $\epsilon_0$ the $S$-matrix can be approximated by Breit-Wigner (BW) model

$$S = e^{2i\theta_n^{(reg)}}\frac{\epsilon-\epsilon_0-\frac{i}{2}\Gamma}{\epsilon-\epsilon_0+\frac{i}{2}\Gamma}\,|n\rangle\langle n|\,+\,S^{(reg)} \tag{3}$$

where $S^{(reg)}$ and $e^{2i\theta_n^{(reg)}}$ describe the "regular" parts of the $S$-matrix which have no singularity at $\epsilon = z_j^*$.

(ii) Transport through a quantum dot can be schematically described by a 1D Schroedinger equation with potential that confines particle between two barriers. As a simple model, we consider $U(x) = \lambda\delta(x-a/2) + \lambda\delta(x+a/2)$. In this potential, the resonances $\epsilon_{1,2,3\ldots}$ are described by the standing wave condition $k_n a \approx \pi n$. Consider the limit of strong barriers (large $\lambda$) and find the Breit-Wigner form of the S-matrix for near a resonance $\epsilon_n$.

Find the transmission and reflection coefficients, $T(\epsilon) = |t|^2$ and $R(\epsilon) = |r|^2$, and analyze their energy dependence near a resonance.

**3. Quasi-bound states and scattering for inverted parabola.** *(This is a hard one)*

In this problem we analyze scattering on an inverted parabola, used in Problem 1 to model quantum point contact. Consider a 1d Schrödinger equation

$$\epsilon\psi = -\frac{1}{2}\psi'' - \frac{1}{2}x^2\psi \tag{4}$$

We learned in Problem 2 that complex poles and zeros are related to quasi-bound states which have only the "out" but no "in" component. Here we demonstrate how one can reconstruct the $S$ matrix by analyzing these states.

(i) Show that the state $\psi_0(x) = e^{ix^2/2}$ represents a quasi-bound state for complex energy $\epsilon_0 = -i/2$ which has only outgoing but no incoming component at infinity.

Write the Hamiltonian in the form $H = -a^\dagger a - i/2$, where

$$a = 2^{-1/2}(x + id/dx), \quad a^\dagger = 2^{-1/2}(x - id/dx)$$

are analogs of the ladder (lowering and raising) operators for this problem. From the commutators of these operators and $H$, treating $a$, $a^\dagger$ by analogy with harmonic oscillator problem, show that there are quasi-energy states with energies

$$\epsilon_n = -i\left(n + \frac{1}{2}\right), \quad n \geq 0, \tag{5}$$

where even and odd values of $n$ correspond to even and odd states $\psi_n(x)$.

(ii) Show that the $2 \times 2$ scattering matrix for this problem can be diagonalized in the channel space in the even and odd basis:

$$S = e^{2i\theta_+}|+\rangle\langle+| + e^{2i\theta_-}|-\rangle\langle-|, \quad \text{where} \quad |+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}, \tag{6}$$

and $\theta_\pm$ are energy-dependent scattering phases for the even and odd channels. From this representation, find the ratio of the transmission and reflection amplitudes,

$$f(\epsilon) = t(\epsilon)/r(\epsilon) \tag{7}$$

in terms of the phase factors $e^{2i\theta_\pm}$.

(iii) Use the information obtained in part (i), combined with the result of Problem 1, to show that the quantities $e^{2i\theta_+}$ ($e^{2i\theta_-}$) as a function of complex energy, have zeros at $\epsilon = -\epsilon_n$ and poles at $\epsilon = \epsilon_n$, where $n \geq 0$ is an even (odd) integer. Use this result to evaluate the function $f(\epsilon)$ at the points $\epsilon = \pm i(n + 1/2)$ of the complex $\epsilon$ plane.

Having this information, reconstruct the function $f(\epsilon)$ and to find the transmission and reflection coefficients for inverted parabola $U(x) = -\frac{1}{2}m\omega^2 x^2$.

*Hint:* Two analytic functions of complex variable which coincide on a set of points $z_n$ with a limiting point somehere in the complex plane or at infinity, coincide in the entire plane.