

Localized and delocalized states in one dimension

1. Tight-binding model with random potential

Consider a discrete Schrodinger equation,

$$\epsilon\psi_n = V_n\psi_n + \psi_{n-1} + \psi_{n+1}, \quad 0 < n < N,$$

with potential V_n taking random values, uniformly distributed in the interval $[-W, W]$.

As discussed in class, one can construct a 2×2 transfer matrix relating variables on the left and on the right:

$$\begin{pmatrix} \psi_N \\ \psi_{N-1} \end{pmatrix} = M \begin{pmatrix} \psi_1 \\ \psi_0 \end{pmatrix}, \quad M = \prod_{n=1 \dots N-1} G_n, \quad G_n = \begin{pmatrix} \epsilon - U_n & -1 \\ 1 & 0 \end{pmatrix}$$

The winding number of the trajectory in the 2d plane parameterized by the wavefunction amplitudes (ψ_N, ψ_{N-1}) , taken as a function of energy, is related to the number of eigenstates, whereas its logarithmic growth (the Lyapunov exponent), gives the inverse localization length:

$$\gamma(\epsilon) + i\pi N(\epsilon) = \lim_{N \rightarrow \infty} \frac{1}{N} \log(\psi_N + i\psi_{N-1}) \tag{1}$$

a) First, consider a clean system, $U_n = 0$. Find the transfer matrix M and check that the formula (1) makes sense. Compare with the solution of the tight-binding model (see Problem 1, PS#1). Do your $\gamma(\epsilon)$ and $N(\epsilon)$ satisfy the Thouless relation, $\gamma(\epsilon) = P \int N(\epsilon') \frac{d\epsilon'}{\epsilon' - \epsilon}$? (In other words, is the function $\gamma(\epsilon) + i\pi N(\epsilon)$ analytic?)

b) For a disordered system, compute the transfer matrix numerically, and use it to find the Lyapunov exponent $\gamma(\epsilon)$ and the density of states. Use parameter values $W = 0.5, 1, 2$, and N of a few tens. Are the results improved when N is increasing?

2. Harper equation, duality, localization transition

In a tight-binding problem with a quasiperiodic potential,

$$\epsilon\psi_n = 2t' \cos(2\pi\omega n + \theta) \psi_n + t\psi_{n-1} + t\psi_{n+1} \tag{2}$$

the eigenstates can be either localized or delocalized depending on the ratio of t and t' . There is an Anderson transition when $t = t'$.

To understand the origin of this behavior, let us consider Fourier-transformed wavefunction, $\psi_n = \int_{-\pi}^{\pi} \frac{dp}{2\pi} \psi_p e^{ipn}$, and rewrite the Schrodinger equation for ψ_p . Taking into account that shift $n' = n \pm 1$ translates into multiplication by a phase factor $\psi_p \rightarrow e^{\pm ip} \psi_p$, and conversely, the Fourier transform of $2 \cos(2\pi\omega n + \theta) \psi_n$ is $e^{i\theta} \psi_{p+2\pi\omega} + e^{-i\theta} \psi_{p-2\pi\omega}$, we write

$$\epsilon\psi_p = t' \psi_{p+2\pi\omega} + t' \psi_{p-2\pi\omega} + 2t \cos(p) \psi_p$$

where without loss of generality we set $\theta = 0$. After rescaling, $p = 2\pi\omega\tilde{p}$, we find

$$\epsilon\psi_p = 2t \cos(2\pi\omega\tilde{p}) \psi_{\tilde{p}} + t' \psi_{\tilde{p}-1} + t' \psi_{\tilde{p}+1}$$

From this, we can argue that there is localization when $|t'| > |t|$ and delocalization when $|t'| < |t|$.

a) Show that this is true by solving the problem numerically. Which approach, the transfer matrix method described in Problem 1, or direct diagonalization of the Hamiltonian for a finite system, works better?

b) The famous Hofstadter Butterfly is what you get for the density of states at the critical point, $|t'| = |t|$ (see lecture notes). It is a fractal set with intricate structure. How detailed a butterfly can you draw with a limit of 1/2 hour for CPU run time?