Theory of Random Matrices

1. Spectral statistics and level repulsion for random matrices.

a) Consider 2×2 hermitian matrices H drawn from a Gaussian ensemble,

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}, \quad h_{21} = h_{12}^*, \quad dP(H) = ce^{-a\mathrm{Tr}\, H^2} dh_{11} dh_{22} dh'_{12} dh''_{12}$$

where $h_{12} = h'_{12} + ih''_{12}$. Find the joint distribution function of the eigenvalues of H.

b) Same as in a) but for 2×2 real-valued hermitian matrices (i.e. symmetric real matrices, $\beta = 1$). Compare with the result found in part a) for nearly conciding energy levels, and show that level repulsion is weaker than for hermitian matrices, $\beta = 2$.

2. Density of states for Gaussian matrices.

Consider the density of states for a Gaussian ensemble of $N \times N$ hermitian matrices, $P(H) \propto e^{-a\mathrm{Tr}\, H^2}$.

a) By using a self-consistent Born approximation for the Greens function $G(\epsilon) = 1/(\epsilon - H + i\delta)$, or otherwise, show that the ensemble-averaged density of states at large N is given by a semicircle $\rho(\epsilon) = \Delta^{-1}\sqrt{1 - \epsilon^2/\epsilon_0^2}$.

b) Relate the values of Δ and ϵ_0 to N and a.

3. Spectral correlation function at large N (hermitian ensemble).

a) Consider the ground state $\psi(x_1, ..., x_N)$ of N noninteracting fermions moving in a one-dimensional parabolic potential. Show that the curvature of the potential can be chosen so that the probability $|\psi(x_1, ..., x_N)|^2$ coincides with the eigenvalue distribution in a Gaussian ensemble of hermitian random matrices, $\beta = 2$, whereby the Pauli exclusion principle mimics level repulsion of random matrices.

b) Consider a one-dimensional ideal Fermi gas with a constant, spatially unform density n. Find the two-point density correlator $\langle \rho(x)\rho(x') \rangle = \langle \psi^\dagger(x)\psi(x)\psi^\dagger(x')\psi(x') \rangle$. Use this result to obtain the pair correlation function of energy levels for the hermitian ensemble.