1. Bound states in 1D.
   a) Consider 1D Schrödinger equation with delta-function scattering potential, \( U(x) = \alpha \delta(x) \). Show that a localized eigenstate can exist attractive potential (\( \alpha < 0 \)), such that \( |\psi(x)| \) decreases exponentially at large \( x \to \pm \infty \). What is the energy of this state?
   
   b) Consider a tight-binding model in 1D described by the nearest neighbor hopping Hamiltonian:
   \[
   H = \sum_n t (|n\rangle \langle n+1| + |n+1\rangle \langle n|) + V_n |n\rangle \langle n|,
   \]
   where \( t \) is the hopping amplitude and the potential \( V_n \) is nonzero only on one site:
   \[
   V_{n=0} = \epsilon_0, \quad V_{n\neq0} = 0.
   \]
   Show that in this problem there are extended eigenstates with all energies in the interval \(-2t < \epsilon < 2t\) and at most one localized state with energy outside this interval.

2. Coexistence of localized and extended states.
   a) For the 1D Shrödinger equation \(-\frac{\hbar^2}{2m} \psi'' + U(x) \psi = \epsilon \psi\) with an arbitrary \( U(x) \), show that a localized state which decreases exponentially at infinity cannot occur at the same energy as an extended state which asymptotically behaves as a superposition on several plane waves.
   
   In this problem it is useful to use the Wronskian defined as \( W(x) = \tilde{\psi}_1(x) \psi_2'(x) - \tilde{\psi}_1'(x) \psi_2(x) \). The quantity \( W(x) \) has the property of being constant (i.e. \( x \)-independent) for any two solutions \( \psi_{1,2}(x) \) with the same energy \( \epsilon \). (Can you prove it?)
   
   b) Generalize the result of part a) to the Shrödinger equation in arbitrary dimension \( D \), and thereby prove Mott’s theorem that localized and extended states cannot coexist. In a generic random potential these two types of states occur at different energies, separated by a threshold called “mobility edge.”

   a) For the tight-binding model with a localized scatterer introduced in Problem 1 b) define in- and out-states and find the \( S \)-matrix.
   
   b) Verify that the \( S \)-matrix is unitary.
   
   c) Find transmission and reflection coefficients, \( T = |t|^2 \), \( R = |r|^2 \), and the scattering phases \( \theta_{1,2} \) given by \( e^{2i\theta_{1,2}} \), the eigenvalues of \( S \).