8.513: QUANTUM TRANSPORT Problem Set # 1 Due: **09/11/08**

Localized and delocalized states. Scattering approach

1. Bound states in 1D.

a) Consider 1D Schrödinger equation with delta-function scattering potential, $U(x) = \alpha\delta(x)$. Show that a localized eigenstate can exist attractive potential ($\alpha < 0$), such that $|\psi(x)|$ descreases exponentially at large $x \to \pm\infty$. What is the energy of this state?

b) Consider a tight-binding model in 1D described by the nearest neighbor hopping Hamiltonian:

$$H = \sum_n t\,(|n\rangle\langle n+1| + |n+1\rangle\langle n|) + V_n|n\rangle\langle n|,$$

where t is the hopping amplitude and the potential V_n is nonzero only on one site:

$$V_{n=0} = \epsilon_0, \quad V_{n\neq 0} = 0.$$

Show that in this problem there are extended eigenstates with all energies in the interval $-2t < \epsilon < 2t$ and at most one localized state with energy outside this interval.

2. Coexistence of localized and extended states.

a) For the 1D Shrödinger equation $-\frac{\hbar^2}{2m}\psi'' + U(x)\psi = \epsilon\psi$ with an arbitrary $U(x)$, show that a localized state which decreases exponentially at infinity cannot occur at the same energy as an extended state which asymptotically behaves as a superposition on several plane waves.

In this problem it is useful to use the Wronskian defined as $W(x) = \bar{\psi}_1(x)\psi_2'(x) - \bar{\psi}_1'(x)\psi_2(x)$. The quantity $W(x)$ has the property of being constant (i.e. x-independent) for any two solutions $\psi_{1,2}(x)$ with the same energy ϵ. (Can you prove it?)

b) Generalize the result of part a) to the Shrödinger equation in arbitrary dimension D, and thereby prove Mott's theorem that localized and extended states cannot coexist. In a generic random potential these two types of states occur at different energies, separated by a threshold called "mobility edge."

3. Scattering matrix.

a) For the tight-binding model with a localized scatterer introduced in Porblem 1 b) define in- and out-states and find the S-matrix.

b) Verify that the S-matrix is unitary.

c) Find transmission and reflection coefficients, $T = |t|^2$, $R = |r|^2$, and the scattering phases $\theta_{1,2}$ given by $e^{2i\theta_{1,2}}$, the eigenvalues of S.