8.513 **11/25/2008 (by Nan Gu)**

One-Dimensional Models of Disordered Systems

Theory on one-dimensional disordered systems, which we shall discuss today, historically preceeded the developments we talked about earlier in the course (Anderson localization, weak localization, mesoscopic phenomena, etc.). Compared to the latter problems, one-dimensional problems afford a larger variety of mathematical tools. Furthermore, quite often an exact solution can be constructed.

We first discuss the density of states (DOS), defined as $\rho(\varepsilon) = \frac{1}{\pi}\,\mathrm{Im}\,(\varepsilon - H + i0)^{-1}$, which is the simplest mathematical quantity associated with a disordered system. There are several techniques available, which can yieled exact results for DOS in many 1d problems. Furthermore, as we shall see, density of states can be related to localization length. Knowing one as a function of energy will give the other.

We shall start with 1d Schrödinger Equation with random potential,

$$-\psi'' + U(x)\psi = \varepsilon\psi \tag{1}$$

(it will be convenient to use units in which $\hbar^2/2m \equiv 1$).

If the potential U is zero, the Schrödinger operator (1) has continuous spectrum at positive energies. Thus $\rho(\varepsilon)$ is zero at negative ε, whereas $\rho(\varepsilon > 0) \propto \varepsilon^{-1/2}$.

One interesting example of the potential U, for which the density of states can be found exactly, is a sum of randomly placed delta-functions, $U(x) = \sum_i \delta(x - x_i)$, with positions x_i uncorrelated and having uniform spatial density c. In this case, the density of states deviates from that in a clean system at the energies $\varepsilon \lesssim c^2$, as illustrated in Fig.1.

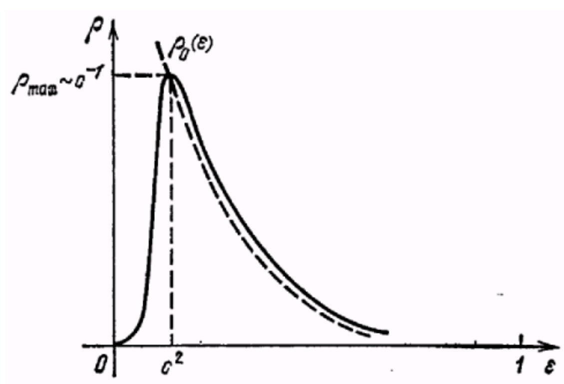

Figure 1: Density of states $\rho(\varepsilon)$ for random potential $U(x) = \sum_i \delta(x - x_i)$, with spatial concentration of the delta functions equal to c. The density of states of a clean system is labeled $\rho_0(\varepsilon)$.

Phase Method

The idea of this method comes from a general relation between the number of zeros of a wavefunction and the energy levels, as given, for example, by the Sturm-Liouville theorem. Given a wavefunction $\psi(0 < x < L)$, obtained by solving the Schrodinger equation at some energy ε with a boundary condition $\psi'(0) = a\psi(0)$ (with a real), we can use the number of sign changes of $\psi(x)$ in the interval $0 < x < L$ to estimate the number of energy levels with energies below ε. The number of states in the spectrum with $\varepsilon' < \varepsilon \approx$ the number of zeros of the wavefunction $\psi(\varepsilon)$ (up to ± 1, depending on the values of a in the boundary cnditions)).

Re-write Schrodinger equation as a first order differential equation:

$$\phi = \begin{pmatrix} \psi \\ \psi' \end{pmatrix} \qquad \frac{d\phi}{dx} = \begin{pmatrix} 0 & 1 \\ U(x) - \varepsilon & 0 \end{pmatrix} \phi$$

Write: $w = \psi + i\psi'$, $\cot\alpha = \frac{\psi'}{\psi}$ ($\alpha = \frac{\pi}{2} - \arg(\psi + i\psi')$).

$$\alpha' = \Phi_\varepsilon(U, \alpha) \quad \Phi_\varepsilon(U, \alpha) = \cos^2\alpha + (\varepsilon - U(x))\sin^2\alpha$$

From this one can show that α is growing, $\frac{\partial\alpha}{\partial\varepsilon} > 0$.

$$N(\varepsilon_1 < \varepsilon < \varepsilon_2) = \lim_{L\to\infty} \frac{1}{\pi L}\left(\alpha(L,\varepsilon_2) - \alpha(L,\varepsilon_1)\right)$$

White Noise model:

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$$\bar{U}(x) = 0, \langle U(x)U(x')\rangle = 2D\delta(x - x')$$

$$z = \frac{\psi'}{\psi}(= \cot\alpha) \qquad z' = -(z^2 + \varepsilon) + U(x) \qquad \text{Langevin-type Equation}$$

Convert to

Fokker-Planck equation for $P(z,\varepsilon)$:

$$\frac{\partial P(z)}{\partial x} = \frac{\partial}{\partial z}\left((z^2+\varepsilon)P(z) + D\frac{\partial P}{\partial z}\right)$$

Solving for the stationary-state solutions, the quantity in parentheses is dependent on :

$$J(\varepsilon) = (z^2+\varepsilon)P(z) + D\frac{\partial P}{\partial z}$$

This is solved by:

$$\frac{DP(z)}{J(\varepsilon)} = \exp\left(-\frac{z^3}{3D} - \frac{\varepsilon z}{D}\right)\int_{-\infty}^{z} \exp\left(\frac{t^3}{3D} + \frac{\varepsilon t}{D}\right) dt$$

Determine $J(\varepsilon)$ from normalization:

$$N(\varepsilon)? = J(\varepsilon) \qquad J^{-1}(\varepsilon) = D^{-\frac{1}{2}}\int_{-\infty}^{\infty} dx \int_{-\infty}^{x} dy\ \exp\left(\phi(y) - \phi(x)\right) \qquad \phi(x) = \frac{x^3}{3} + \frac{\varepsilon x}{D^{\frac{2}{3}}} \qquad x = \frac{t}{D^{\frac{1}{3}}}$$

This gives an exact form of the disorder-averaged density of states, including the tail at negative energies. The tail arises because in each realization of white noise there are wide regiosn in which potential takes predominantly negative values.

Localization Length and the Lyapunov exponent:

$$\psi(x) = r(x)\cos\alpha(x) \qquad \psi' = r(x)\sin\alpha(x)$$

Lyapunov Exponent:

$$\gamma(\varepsilon) = \lim_{x\to\infty}\frac{\log(r(x))}{x}$$

Transfer Matrix:

$$\phi = \begin{pmatrix} \psi \\ \psi' \end{pmatrix} \qquad \frac{d\phi}{dx} = \begin{pmatrix} 0 & 1 \\ U(x)-\varepsilon & 0 \end{pmatrix}\phi \qquad \phi(x) = M(x)\phi(0) \qquad \frac{dM}{dx} = \begin{pmatrix} 0 & 1 \\ U(x)-\varepsilon & 0 \end{pmatrix} M$$

Formal Solution:

$$M(x) = L\exp\left(\int_0^L \begin{pmatrix} 0 & 1 \\ U(x')-\varepsilon & 0 \end{pmatrix}\right) dx'$$

Eigenvalues of M: m, $\frac{1}{m}$, $m > 0$. $\det M = 1$

$$m, \frac{1}{m} = \exp\left(\pm 2x_i\right)$$

(see last lecture). Thus the localization length is given by inverse Lyapunov exponent, $\xi = 1/\gamma$.

$$\gamma(\varepsilon) = \lim_{x\to\infty}\frac{\log(m)}{x}$$

For a tight-binding model:

$$\varepsilon\psi_n = U_n\psi_n + \psi_{n-1} + \psi_n \qquad \phi = \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix} \qquad \psi_{n+1} = \begin{pmatrix} \varepsilon - U_n & -1 \\ 1 & 0 \end{pmatrix}\phi_n$$

Thouless theorem:

There is a very general result due to Thouless, which gives an explicit relation between the density of states and Lyapunov exponent:

$$\gamma(\varepsilon) = \int \log|\varepsilon - \varepsilon'|\rho(\varepsilon')d\varepsilon'$$

To prove this result, let us combine our formulas for the number of states, written as a winding number, and the Lyapunov exponent, written as a logarithmic growth rate, into one equation:

$$\gamma(\varepsilon) + i\pi N(\varepsilon) = \lim_{n\to\infty} \frac{1}{n} \log(\psi_n + i\psi_{n+1})$$

The quantity on the right hand side, $\psi_n + i\psi_{n+1}$, is an analytic function of ε, $\mathrm{Im}(\varepsilon) > 0$ (also true at $\mathrm{Im}(\varepsilon) < 0$). The anayticity, which is a property of the wavefunction ψ_n found from Shrodinger equation for specific realization of disorder, holds also after averaging over disorder.

We are going to use Cauchy theorem, which gives an integral representation of an analytic funcion in terms of an integral taken over the boundary of the analyticity domain,

$$f(\varepsilon) = \int \frac{f(\varepsilon')}{\varepsilon' - \varepsilon + i0^+} \frac{d\varepsilon'}{2\pi i}$$

In our case, with $f(\varepsilon) = \langle \gamma(\varepsilon) + i\pi N(\varepsilon)\rangle_{\mathrm{dis}}$, the integral, which is taken over $-\infty < \varepsilon' < \infty$, can be written as

$$f(\varepsilon) = \frac{1}{2} f(\varepsilon) + P\int \frac{f(\varepsilon')}{\varepsilon' - \varepsilon} \frac{d\varepsilon'}{2\pi i} \qquad f(\varepsilon) = \frac{1}{i\pi} P\int \frac{f(\varepsilon')}{\varepsilon' - \varepsilon} d\varepsilon'$$

where we used the identity $\frac{1}{\varepsilon' - \varepsilon - i0} = \frac{1}{\varepsilon' - \varepsilon} + i\pi\delta(\varepsilon' - \varepsilon)$. Isolating the real and imaginary parts, this gives

$$\gamma(\varepsilon) = \frac{1}{\pi} P\int \frac{\pi N(\varepsilon)}{\varepsilon' - \varepsilon} d\varepsilon' = P\int \log|\varepsilon' - \varepsilon|\rho(\varepsilon')d\varepsilon'$$

where we integrated by parts by writing $\rho(\varepsilon) = dN(\varepsilon)/d\varepsilon$.

Homework Problem:

In a tight-binding problem with a quasiperiodic potential,

$$\varepsilon\psi_n = 2t'\cos(2\pi\omega n + \theta)\,\psi_n + t\psi_{n-1} + t\psi_{n+1} \tag{2}$$

the eigenstates can be either localized or delocalized depending on the ratio of t and t'. There is an Anderson transition when $t = t'$.

To understand the origin of this behavior, let us consider Fourier-transformed wavefunction, $\psi_n = \int_{-\pi}^{\pi} \frac{dp}{2\pi} \psi_p e^{ipn}$, and rewrite the Schrodinger equation for ψ_p. Taking into account that shift $n' = n \pm 1$ translates into multiplication by a phase factor $\psi_p \to e^{\pm ip}\psi_p$, and conversely, the Fourier transform of $2\cos(2\pi\omega n + \theta)\,\psi_n$ is $e^{i\theta}\psi_{p+2\pi\omega} + e^{-i\theta}\psi_{p-2\pi\omega}$, we write

$$\varepsilon\psi_p = t'\psi_{p+2\pi\omega} + t'\psi_{p-2\pi\omega} + 2t\cos(p)\psi_p$$

where without loss of generality we set $\theta = 0$.

After rescaling, $p = 2\pi\omega\tilde{p}$, we find

$$\varepsilon\psi_p = 2t\cos(2\pi\omega\tilde{p})\psi_{\tilde{p}} + t'\psi_{\tilde{p}-1} + t'\psi_{\tilde{p}+1}$$

From this, we can argue that there is localization when $|t'| > |t|$ and delocalization when $|t'| < |t|$. There is a transition at $|t| = |t'|$.

Analyze this numerically. The famous Hoffstadter Butterfly is what you get at $|t'| = |t|$ (see Fig.2).

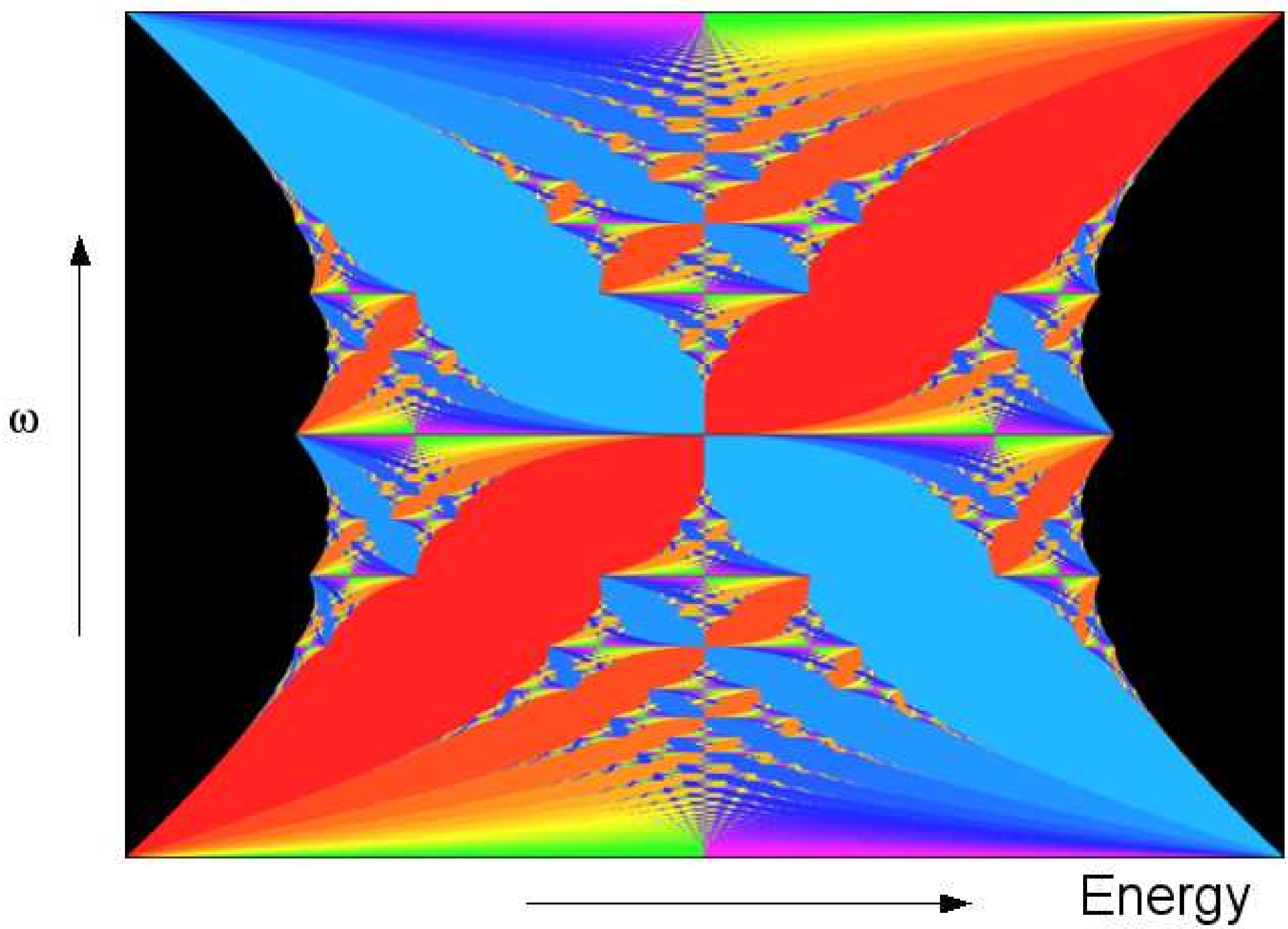

Figure 2: Density of states $\rho(\varepsilon)$ for quasiperiodic tight-binding problem (2), with $0 < \omega < 1$.