Lectures 5,6,7: Boltzmann kinetic equation

Sep 18, 23, 25, 2008

Fall 2008 8.513 "Quantum Transport"

- Distribution function, Liouville equation
- Boltzmann collision integral, general properties
- Irreversibility, coarse graining, chaotic dynamics
- Relaxation of angular harmonics
- Example: Drude conductivity
- Diffusion equation
- Magnetotransport
- Quantizing fields: Shubnikov-de Haas oscillations
- Transport in a smooth, long-range-correlated disorder

- Calculations of magnetoresistance
- Weiss oscillations
- Lorentz model; generalized Boltzmann equation

Scattering by randomly placed impurities

Disorder potential $V(r) = \sum_j V(r - r_j)$. B.k.e. takes the form

$$\frac{df}{dt} \equiv Lf = \int w(p,p')(f(p') - f(p))\frac{d^2p'}{(2\pi\hbar)^2}$$

where $L = \partial_t + \mathbf{v}\nabla_r + \mathbf{F}\nabla_p$ is the Liouville operator.

In the Born approximation $w(p,p') = 2\pi|V_{p-p'}|^2\delta(\epsilon_p - \epsilon_{p'})$. Writing $\int \frac{d^2p'}{(2\pi\hbar)^2} = \oint \frac{d\theta_{p'}}{2\pi}\int \nu d\epsilon_{p'}$ with the density of states $\nu = dN/d\epsilon = m/(2\pi\hbar^2)$ find

$$\frac{df}{dt} = \nu \int d\theta_{p'}|V_{p-p'}|^2(f(p') - f(p))$$

where $|p'| = |p|$.

Relaxation of angular harmonics

For spatially uniform system ($L = \partial_t$) analyze the angular dependence of the collision integral in B.k.e.

$$\frac{df}{dt} \equiv Lf = \nu \oint \frac{d\theta_{p'}}{2\pi} w(\theta_p - \theta_{p'})(f(p') - f(p))$$

Use the Fourier series

$$f(p) = \sum_m e^{im\theta} \tilde{f}_m, \quad \nu w(\theta_p - \theta_{p'}) = \sum_m \gamma_m e^{im(\theta_p - \theta_{p'})}$$

where $\tilde{f}_0 = \oint f(p') \frac{d\theta_{p'}}{2\pi}$ is proportional to the total particle number, $\tilde{f}_{\pm 1} = \oint e^{\mp i\theta_{p'}} f(p') \frac{d\theta_{p'}}{2\pi}$ are proportional to the particle current density components $j_x \pm ij_y$, etc. For the collision integral we have

$$St(\tilde{f}_m) = (\gamma_m - \gamma_0)\tilde{f}_m, \quad \tilde{f}_m(t) \propto e^{-(\gamma_0 - \gamma_m)t}$$

Relaxation for $m \neq 0$ because $\gamma_{m\neq 0} < \gamma_0$; no relaxation for $\tilde{f}_0$ (particle number conservation)

Example: Drude conductivity

For a spatially uniform system the Boltzmann kinetic equation $(\partial_t + \mathbf{v}\nabla_r + e\mathbf{E}\nabla_p) f(\mathbf{r}, \mathbf{p}) = St(f)$ becomes

$$e\mathbf{E}\nabla_p f(\mathbf{p}) = \mathrm{St}(f)$$

Solve for the perturbation of the distribution function due to the E field:

$$f(\mathbf{p}) = f_0(\mathbf{p}) + f_1(\mathbf{p}) + ..., \quad f_1 \propto \cos\theta, \sin\theta$$

where f_0 is isotropic. To the lowest order in E find

$$f_1 = -\tau_{tr} e\mathbf{E}\nabla_p f_0(\mathbf{p})$$

$$\mathbf{j} = \int e\mathbf{v} f_1(p) \frac{d^2p}{(2\pi\hbar)^2} = \frac{e^2\tau_{tr}}{m}\mathbf{E}\int f_0(p)\frac{d^2p}{(2\pi\hbar)^2} = \frac{e^2\tau_{tr} n}{m}\mathbf{E}$$

We integrated by parts assuming energy independent τ_{tr}, valid for degenerate Fermi gas $T \ll E_F$. At finite temperatures, and for energy-dependent τ_{tr}, find $\sigma = \frac{e^2 n}{m}\langle\tau_{tr}(E)\rangle_E$.

Diffusion equation

Solve B.k.e. for weakly nonuniform density distribution (and no external field!).

$$(\partial_t + \mathbf{v}\nabla_r)\, f(\mathbf{r},\mathbf{p}) = St(f)$$

First, integrate over angles θ_p to obtain the continuity equation:

$$\frac{\partial n}{\partial t} + \nabla \mathbf{j} = 0, \quad \mathbf{j} = \langle \mathbf{v} f \rangle_\theta, \quad n = \langle f \rangle_\theta$$

Here $\mathbf{j}$ and n is particle number current and density. Next, use angular harmonic decomposition $f(\mathbf{p}) = f_0(\mathbf{p}) + f_1(\mathbf{p}) + ...$, $f_1 \propto \cos\theta, \sin\theta$ and relate f_1 with a gradient of f_0. Perturbation theory in small spatial gradients:

$$\mathbf{v}\nabla_r f_0 = -\frac{1}{\tau_{tr}} f_1, \quad \frac{1}{\tau_{tr}} = \langle w(\theta)(1-\cos\theta)\rangle_\theta$$

Use $f_1 = -\tau_{tr}\mathbf{v}\nabla_r f_0$ to find the current $j_\alpha = \langle v_\alpha f_1 \rangle = -\tau_{tr}\langle v_\alpha v_\beta\rangle \nabla_\beta f_0$. From $\langle v_\alpha v_\beta \rangle = \frac{1}{2}\delta_{\alpha\beta} v_F^2$ have

$$\mathbf{j} = -\frac{1}{2}\tau_{tr} v_F^2 \nabla f_0 = -D\nabla f_0 = -D\nabla n$$

Features

Diffusion constant

$$D = \frac{1}{2}\tau_{tr} v_F^2 = \frac{1}{2} v_F \ell, \quad \ell = v_F \tau_{tr}$$

where ℓ is the mean free path. From Einstein relation $\sigma = e^2 \nu D$ find

$$\sigma = e^2 \nu D = g_s g_v \frac{e^2}{h} k_F \ell = \frac{e^2 \tau_{tr} n}{m}$$

where $g_{s,v}$ the spin and valley degeneracy factors.

Used the density of states $\nu = g_s g_v \frac{2\pi p dp}{(2\pi\hbar)^2 dE} = \frac{g_s g_v k_F}{2\pi\hbar v_F}$

1) Temperature dependence due to $\tau_{tr}(E)$, weak near degeneracy;

2) Boltzmann eqn. is a quasiclassical treatment valid for $k_F \ell \gg 1$. In this case $\sigma \gg \frac{e^2}{h}$, metallic behavior;

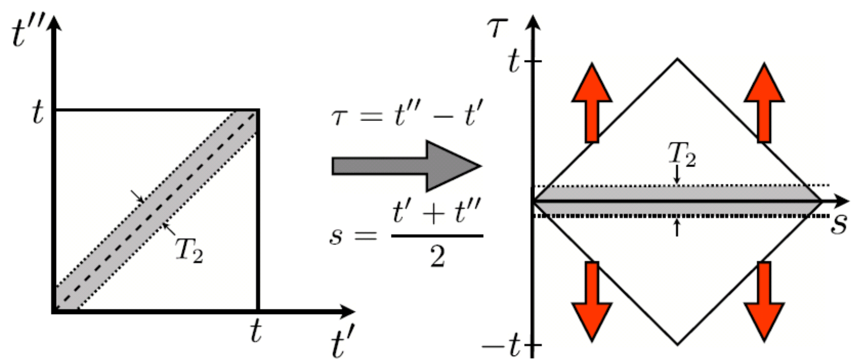
3) At $k_F l \sim 1$, or $\lambda_F \sim \ell$, onset of localization

Another approach

Diffusion constant from velocity correlation:

$$D = \int_0^\infty dt \langle v_x(t) v_x(0) \rangle, \quad \text{ensemble averaging} : \theta,\ \text{disorder}$$

Derivation: $\langle (x(t) - x(0))^2 \rangle = \int_0^t \int_0^t dt' dt'' \langle v_x(t') v_x(t'') \rangle = 2Dt$.
Explanation ($T_2 \equiv \tau_{tr}$):



Exponentially decreasing correlations $\langle v_x(t) v_x(0) \rangle = e^{-t/\tau_{tr}} \langle v_x^2(0) \rangle = \frac{1}{2} v_F^2 e^{-t/\tau_{tr}}$ yield $D = \frac{1}{2} v_F^2 \tau_{tr}$.

Magnetotransport

Finite $\mathbf{B}$ field, Lorentz force, current not along $\mathbf{E}$. Thus conductivity σ not a scalar but a 2x2 tensor.

Einstein relation $\sigma_{\alpha\beta} = e^2 \nu D_{\alpha\beta}$ for the diffusion tensor

$$D_{\alpha\beta} = \int_0^\infty dt \langle v_\alpha(t) v_\beta(0) \rangle, \quad \alpha, \beta = 1, 2$$

Between scattering events circular orbits $\omega_c = eB/m$, $R_c = mv_F/eB$. Complex number notation $\tilde{v}(t) = v_x(t) + iv_y(t) = v_F e^{i\theta + i\omega_c t}$. Find D from

$$D_{xx} + iD_{yx} = \int_0^t dt \langle \tilde{v}(t) \cos\theta v_F \rangle_\theta e^{-t/\tau_{tr}} = \frac{D}{1 + (\omega_c \tau)^2} (1 - i\omega\tau)$$

$D_{yy} = D_{xx}$, $D_{xy} = -D_{yx}$.

Conductivity and resistivity tensors

$$\hat{\sigma} = \frac{\sigma}{1+(\omega_c\tau)^2}\begin{pmatrix} 1 & -\omega_c\tau \\ \omega_c\tau & 1 \end{pmatrix}, \quad \hat{\rho} = \hat{\sigma}^{-1} = \rho\begin{pmatrix} 1 & \omega_c\tau \\ -\omega_c\tau & 1 \end{pmatrix}$$

The off-diagonal element:

$$\rho_{xy} \equiv R_H = \frac{B}{en} = \frac{1}{g_s g_v}\frac{h}{e^2}\frac{\hbar\omega_c}{E_F}$$

Features:

(i) Classical effects of B field important when $\omega_c\tau \gtrsim 1$, these field can be weak in high mobility samples;

(ii) Recover classical Hall resistivity;

(iii) Zero magnetoresistance in this model: $\rho_{xx}(B) - \rho_{xx}(0) = 0$. Generic for short-range scattering.

(iv) The features (i), (ii), (iii) are fairly robust in the classical model, but not in the presence of quantum effects.

Quantum effects

We've used $\sigma = e^2 \nu D$ with the zero-field density of states; assumed that τ_{tr} is independent of B.

From Born approximation for delta function impurities $U(r) = \sum_j u\delta(r - r_j)$ we have (see above):

$$\tau^{-1} = \frac{2\pi}{\hbar} \nu(E_F) u^2 c_i$$

with c_i the impurity concentration. For $\nu(E_F)$ modulated by Landau levels

$$\nu(\epsilon) = \sum_{n>0} n_{\mathrm{LL}} \delta(\epsilon - n\hbar\omega_c), \quad n_{\mathrm{LL}} = B/\Phi_0 = eB/h$$

find Shubnikov-de Haas (SdH) oscillations periodic in $1/B$.

Period found from particle density on one Landau level $n_{\mathrm{LL}} = eB/h$, giving $\Delta(1/B) = \frac{e}{h} \frac{g_s g_v}{n_{el}}$. Can be used to determine electron density.

Plateaus in R_H at $R_H = \frac{1}{g_s g_v} \frac{h}{e^2} \frac{1}{N}$, with $N = 1, 2, \ldots$: the integer Quantum Hall effect.

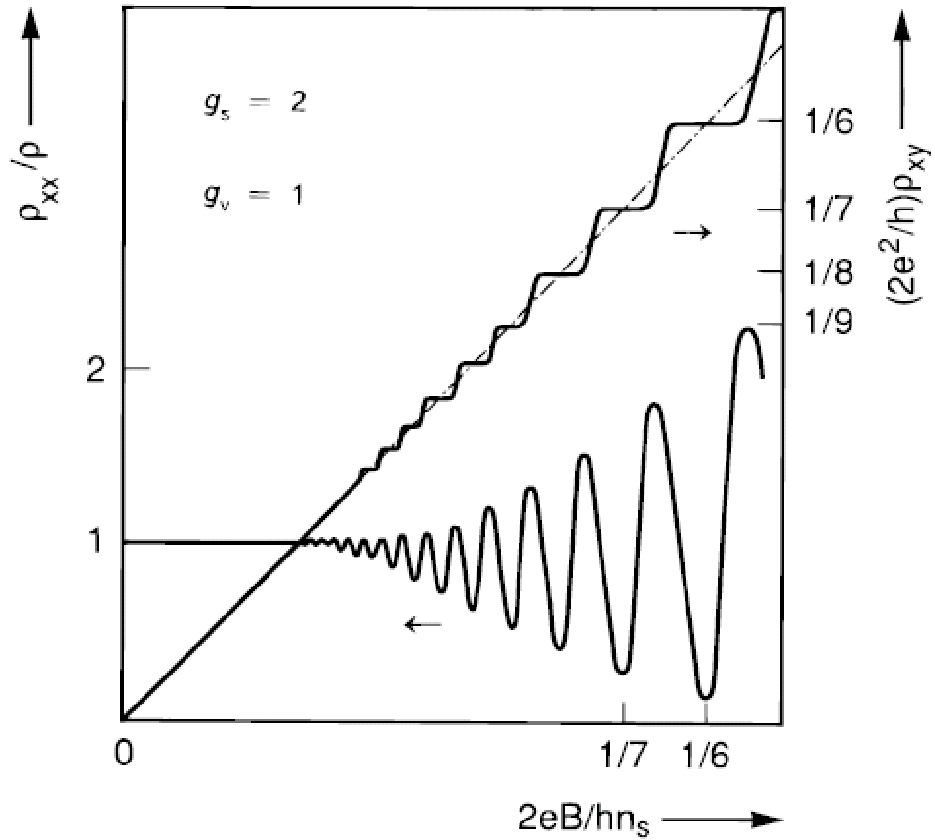


Figure 1: Schematic dependence of the longitudinal resistivity ρ_{xx} (normalized to the zero-field resistivity) and of the Hall resistivity $\rho_{xy} = R_H$ (normalized to $h/2e^2$) on the reciprocal filling factor $\nu^{-1} = 2eB/hn_{el}$ (for $g_s = 2$ and $g_v = 1$). Deviations from the quasiclassical result occur in strong B field, in the form of Shubnikov-de Haas oscillations in ρ_{xx} and quantized plateaus in ρ_{xy}.

Applications of the SdH effect

SdH oscillations in D=3 are sensitive to the extremal cross-sections of Fermi surface, which depend on the orientation of magnetic field w.r.p.t. crystal axes

Thus SdH can, and indeed are, used to map out the Fermi surface 3D shapes.

Fermi surface splitting from SdH oscillations

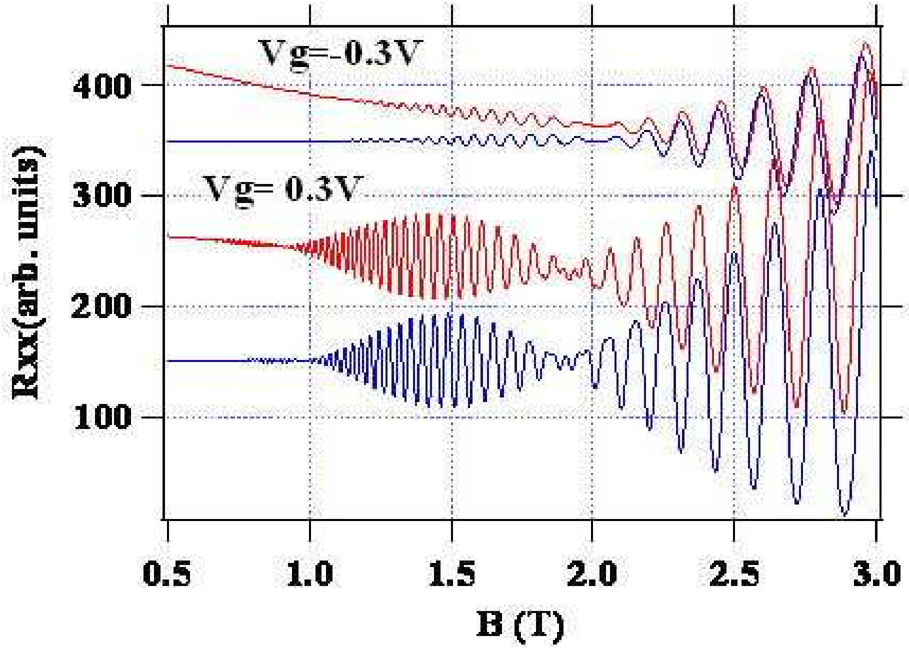


Figure 2: Gate voltage dependence of SdH oscillations. Beating patterns in the SdH oscillations appear due to a spin-orbit interaction. By comparing the oscillations with the numerical simulation based on Rashba spin-orbit interaction, spin-orbit interaction parameter is obtained. From: "Gate control of spin-orbit interaction in an inverted InGaAs/InAlAs heterostructure" J. Nitta, T. Akazaki, H. Takayanagi, and T. Enoki Phys. Rev. Lett., 78, 1335 (1997).

Classical transport in a long-range disorder

Relevant e.g. for high mobility semiconductor systems in which charge donors are placed in a layer at a large distance d from the two-dimensional electron gas (2DEG), $k_F d \approx 10$. Also of pedagogical interest, as an illustration of nonzero magnetoresistance arising from classical transport.

Random potential with long-range correlations, $W(r-r') = \langle V(r)V(r')\rangle$ decays at $|r-r'| \sim d \gg \lambda_F$. It is convenient to introduce the formfactor $\tilde{W}(q) = \int e^{iqr} W(r) d^2 r = \langle |V_q|^2 \rangle$. For a random potential of amplitude $\delta V(r) \sim \alpha E_F$ we estimate $\tilde{W}(q) \approx \tilde{W}(0) \sim (\alpha E_F)^2 d^2$ for $kd \lesssim 1$, and $\tilde{W}(q) \approx 0$ for $kd > 1$.

For charge impurities, as a simple model, one can take $\tilde{W}(q) = \int e^{iqr} W(r) d^3 r = (\pi \hbar^2/m)^2 n_i e^{-2|q|d}$ (the prefactor $(\pi\hbar^2/m)^2$ is due to correlations in impurity positions and/or charge states).

High mobility for $d \gg \lambda_F$. Transport coefficients dominated by small-angle scattering.

Transport time from Fermi's GR

The scattering rate $w_{pp'} = 2\pi |V_{p-p'}|^2 \delta(\epsilon_{p'} - \epsilon_p)$ yields

$$\tau_{tr}^{-1} = \int (1 - \cos\theta') w_{pp'} \frac{d^2 p'}{(2\pi\hbar)^2}$$

Note: the characteristic $p - p' \sim 1/d$, thus $\theta \sim \lambda_F / d \ll 1$.

Using $\int ... \delta(\epsilon_{p'} - \epsilon_p) \frac{d^2p'}{(2\pi\hbar)^2} = \int_{-\pi}^{\pi} \nu \frac{d\theta'}{2\pi}$, expanding $1 - \cos\theta \approx \frac{1}{2}\theta^2$ and replacing $\int_{-\pi}^{\pi} d\theta \ \rightarrow \ \int_{-\infty}^{\infty} d\theta$, find

$$\tau_{tr}^{-1} = \frac{\nu}{2} \int_{-\infty}^{\infty} |V_{p-p'}|^2 \theta^2 d\theta = \frac{\nu \hbar^2}{(m v_F)^3} \int_0^{\infty} \tilde{W}(q) q^2 dq$$

with $\nu = m/(2\pi\hbar^2)$.

Estimate:

Diffusion along the Fermi surface

Slight bending of the classical trajectory (which is a straight line for $V=0$):

$$\frac{dp}{dt}=-\nabla V,\quad \frac{d\theta}{dt}=(mv_F)^{-1}\mathbf{n}\times\nabla V$$

where $\mathbf{n}=\mathbf{v}/|\mathbf{v}|$.

Diffusion in θ:

$$(\partial_t+\mathbf{v}\nabla_r+e\mathbf{E}\nabla_p)\,f(\mathbf{r},\mathbf{p})=D_\theta\frac{\partial^2 f}{\partial\theta^2}\quad (\text{instead of } St(f))$$

where $D_\theta=\int_0^\infty dt\langle\dot\theta(t)\dot\theta(0)\rangle=(mv_F)^{-2}\int_0^\infty\langle\partial_y V(x=v_Ft)\partial_y V(x=0)\rangle=\frac{1}{2m^2v_F^3}\sum_{q_y}q_y^2\tilde W(q_y)$

Derivation from B.k.e.: For the mth harmonic $\tilde f=\oint f(\theta)e^{-im\theta}\frac{d\theta}{2\pi}$ have $St(\tilde f_m)=(\gamma_0-\gamma_m)\tilde f_m$. Approximate: $\gamma_0-\gamma_m=\oint(1-\cos m\theta)w(\theta)\frac{d\theta}{2\pi}\approx\oint\frac{1}{2}m^2\theta^2w(\theta)\frac{d\theta}{2\pi}=D_\theta m^2$. Can expand in θ b/c the integral is dominated by $\theta\sim\lambda_f/d\ll 1$. This gives $St(\tilde f_m)=D_\theta m^2\tilde f_m$, or $St(f(\theta))=D_\theta\partial_\theta^2 f(\theta)$.

Transport time from diffusion eqn

For spatially uniform system write $\partial_t f = D_\theta \frac{\partial^2 f}{\partial \theta^2}$; for the harmonics $\delta f(\theta) \propto \cos\theta, \sin\theta$ have $\delta f(t) \propto e^{-D_\theta t}$. Thus

$$\tau_{tr}^{-1} = D_\theta = \frac{1}{2\pi m^2 v_F^3} \int_0^\infty q^2 \tilde{W}(q) dq$$

which coincides with τ_{tr} found from Fermi's GR.

Magnetotransport problem: memory effects

The relaxation-time approximation: collisions with impurities described by Poisson statistics (no memory about previous collisions): $\langle v_\alpha(t)v_\beta(0)\rangle = e^{-t/\tau}\langle v_\alpha(0)v_\beta(0)\rangle$. In the presence of magnetic field, for complex $v(t) = v_x(t) + iv_y(t)$ have $\frac{1}{2}\langle v(t)v^*(0)\rangle = \frac{1}{2}e^{-t/\tau}e^{i\omega_c t}\langle v(0)v^*(0)\rangle$.

Generalize to a memory function $f(t) = e^{-t/\tau}\left(1 + \sum_{n=1}^{\infty} c_n (t/\tau)^n/n!\right)$. Then the response to a dc electric field $\mathbf{E} \parallel \hat{x}$ will be

$$j_x + ij_y = \frac{ne^2}{m}\int_0^\infty f(t)e^{i\omega_c t}E dt = \frac{\sigma_0 E}{1 - i\omega_c\tau}\left(1 + \sum_{n=1}^{\infty}\frac{c_n}{(1 - i\omega_c\tau)^n}\right)$$

where $\sigma_0 = ne^2\tau/m$ the Drude conductivity. For $c_n = 0$ recover zero magnetoresistance $\Delta\rho_{xx} = \rho_{xx}(B) - \rho_{xx}(0) = 0$ and classical Hall resistivity $\rho_{xy} = B/ne$. However, for a non-Poissonian memory function $f(t)$ the magnetoresistance does not vanish. Thus $\Delta\rho_{xx}(B)$ is a natural probe of the memory effects in transport.

Magnetotransport in a smooth potential

Cyclotron motion in a spatially varying electric field and constant B field:

$$\dot{\mathbf{v}} = \omega_c \hat{z} \times \mathbf{v} + \frac{e}{m}\mathbf{E}(r), \quad \omega_c = eB/m, \quad \mathbf{E}(r) = -\nabla V(r)$$

Drift of the cyclotron orbit guiding center $X = x + v_y/\omega_c$, $Y = y - v_x/\omega_c$. From

$$\dot{v}_x = \omega_c v_y + \frac{e}{m}E_x(r), \quad \dot{v}_y = -\omega_c v_x + \frac{e}{m}E_y(r), \quad r = (x, y)$$

have

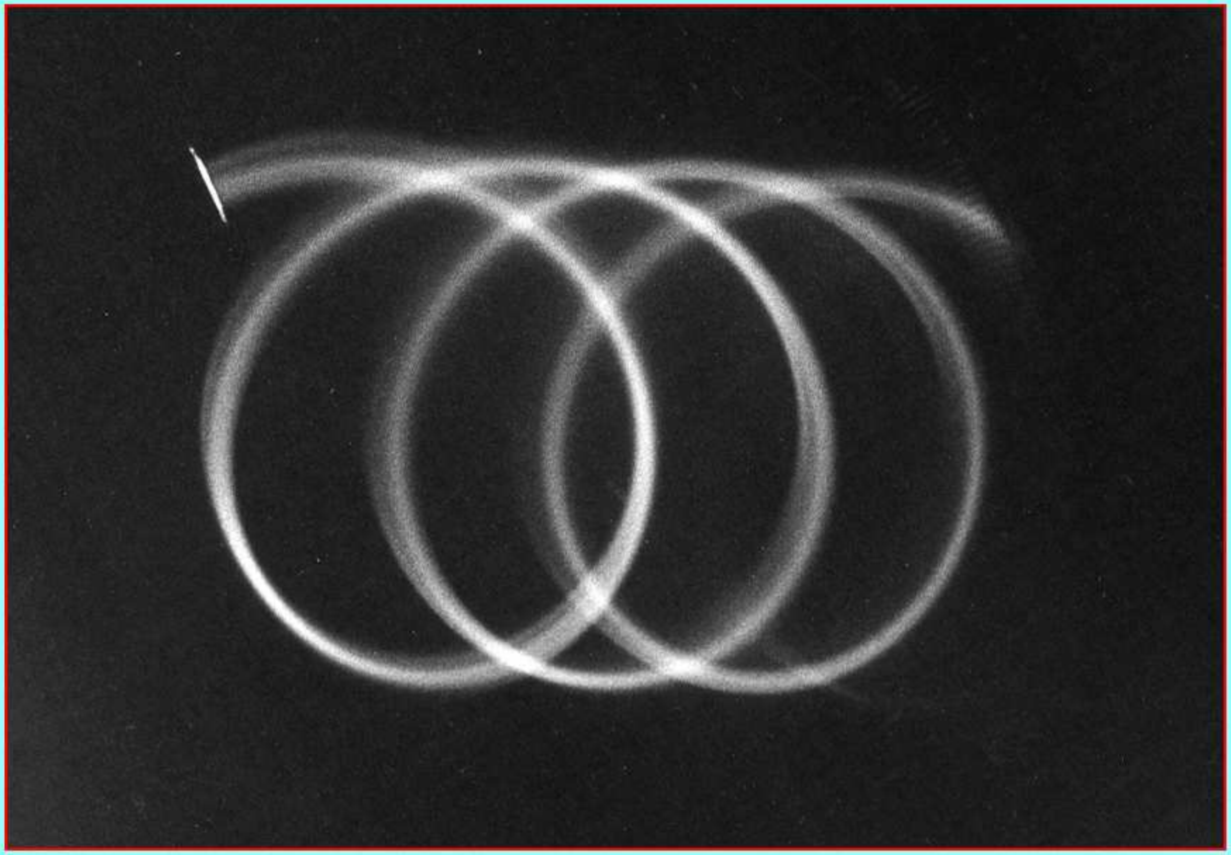
$$\dot{X} = v_x + \dot{v}_y/\omega_c = E_y(r)/B, \quad \dot{Y} = v_y - \dot{v}_x/\omega_c = -E_x(r)/B$$

Find dynamical equation for the guiding center coordinates X and Y:

$$\dot{R} = \frac{1}{B}\mathbf{E}(r) \times \hat{z}, \quad R = (X, Y), \quad r = R + \frac{1}{\omega_c}\hat{z} \times \mathbf{v}$$

equivalent to the Lorentz force equation, useful in the limit $E/B \ll v_F$

Electrons moving in crossed E and B fields



Electron motion in crossed electric and magnetic fields. The trajectory is a cycloid, i.e., a superposition of a circular motion and a constant drift to the right. The cyclotron orbit implies a magnetic field direction into the plane and the $\mathbf{E}\times\mathbf{B}$ drift implies that the electric field points downward. From the known beam energy the field strengths can be obtained from cyclotron radius and guiding center drift.

Taken from http://www.physics.ucla.edu/plasma-exp/beam/

Adiabatic approximation

For $B/E \gg v_F^{-1}$ and slowly varying $E(r)$ have fast cyclotron motion superimposed with slow drift.

Adiabatic approximation: average dynamical quantities (e.g. potential $V(r)$ or the field $\mathbf{E}(r)$) over cyclotron motion with X and Y kept frozen:

$$\langle f(r(t))\rangle_{\rm cycl.motion} = \oint f(X + r_c \sin\phi, Y + r_c \cos\phi)\frac{d\phi}{2\pi}, \quad r_c = v_F/\omega_c$$

(approximate velocity by v_F). Going to Fourier harmonics $f(r) = \sum_q \tilde{f}_q e^{i\mathbf{q}.\mathbf{r}}$, find

$$\langle \sum_q \tilde{f}_q e^{i\mathbf{q}.\mathbf{r}(t)} \rangle_{\rm cycl.motion} = \sum_q \tilde{f}_q e^{i\mathbf{q}.\mathbf{R}(t)} e^{\theta_q} J_0(qr_c), \quad q = \sqrt{q_x^2 + q_y^2}$$

where $J_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\sin\phi} d\phi$ the Bessel function.

Velocity autocorrelation function

$$\mathbf{v} = \dot{\mathbf{R}} + (\dot{\mathbf{r}} - \dot{\mathbf{R}}) = \dot{\mathbf{R}} + \frac{1}{\omega_c}\hat{z} \times \dot{\mathbf{v}} = \text{slow part} + \text{fast part}$$

In the adiabatic approximation, ignoring correlations between the slow part and the fast part, write the autocorrelation function of velocity as

$$\langle v(t)v^*(0)\rangle_{\rm ens} \approx \left(\langle \dot{R}(t)\dot{R}^*(0)\rangle + \frac{1}{\omega_c^2}\langle \dot{v}(t)\dot{v}^*(0)\rangle \right) = e^{-t/\tau}\left(\langle (\dot{R})^2\rangle_{\rm ens} + e^{i\omega_c t} v_F^2\right)$$

The factor $e^{-t/\tau}$ accounts for scattering on the short-range disorder. Substitute the adiabatic-average value

$$\dot{R} = -i\frac{1}{B}\langle E(t)\rangle_{\rm cycl.motion} = \sum_q (q_x + iq_y)\tilde{V}_q e^{i\mathbf{q}.\mathbf{R}(t)} e^{\theta_q} J_0(qr_c)$$

(for complex components ... $\times\,\hat{z} \equiv$ ... $\times\,(-i)$). Treating different harmonics as independent, $\langle V_q V_{q'}\rangle \propto \delta(q+q')$, have

$$\langle (\dot{R})^2\rangle_{\rm ens} = \frac{1}{B^2}\sum_q q^2 |\tilde{V}_q|^2 J_0^2(qr_c)$$

Diffusion from the drift velocity

Find the diffusion tensor components from

$$D_{xx} + iD_{yx} = \int_0^\infty \frac{1}{2}\langle v(t)v^*(0)\rangle_{\text{ens}}dt = \frac{1}{2}\tau\langle(\dot{R})^2\rangle_{\text{ens}} + \frac{v_F^2\tau(1+i\omega_c\tau)}{2(1+\omega_c^2\tau^2)}$$

For an estimate take $\langle|\tilde{V}_q|^2\rangle = \tilde{W}(q) = (\alpha E_F d)^2 e^{-2qd}$, which gives $\sum_q q^2|\tilde{V}_q|^2 J_0^2(qr_c) \sim \frac{(\alpha E_F)^2}{dr_c}$. This is true in the limit of cyclotron radius larger than the correlation length of disorder, $r_c = v_F/\omega_c \gg d$. (The asymptotic form of $J_0(x \gg 1) = \sqrt{\frac{2}{\pi x}}\cos(x - \pi/4)$ was used.)

For a weak short-range disorder (or strong B field) have $\omega_c\tau \gg 1$, giving

$$D_{xx} \approx \frac{(\alpha E_F)^2\tau}{2dr_c B^2} + \frac{v_F^2}{2\omega_c^2\tau} \approx \frac{(\alpha E_F)^2\tau\omega_c}{2dv_F B^2}, \quad D_{yx} \approx \frac{v_F^2}{2\omega_c}$$

Thus both D_{xx} and D_{yx} scale as $1/B$ (same true for σ_{xx} and σ_{yx}). This gives resistivity $\rho_{xx} = \sigma_{xx}/(\sigma_{xx}^2 + \sigma_{yx}^2)$ linear in B at strong fields, $\omega_c\tau \gg 1$.

Note: increase in σ_{xx} implies increase in ρ_{xx} b/c $\rho_{xy} \gg \rho_{xx}$ at strong fields, and thus $\rho_{xx} \approx \sigma_{xx}/\sigma_{xy}^2$. Thus a positive magnetoresistance.

All that can be done more rigorously using B.k.e., see Beenakker, PRL 62, 2020 (1989)

The simple limit considered here is when $\lambda_F \ll d \ll r_c \ll \ell$. In other regimes the analysis can be carried out using Boltzmann equation (see papers by Mirlin, Wölfle, Polyakov et al.)

Linear magnetoresistance in a high mobility 2DEG

Z.D. Kvon et al. / Physica E 22 (2004) 332–335

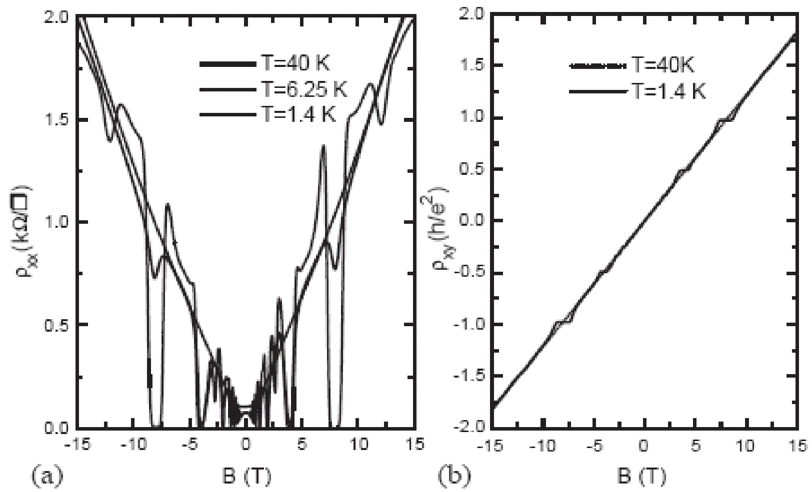


Fig. 1. Dissipative $\rho_{xx}(B)$ (a) and Hall resistivity $\rho_{xy}(B)$ (b) at different temperatures (sample 218).

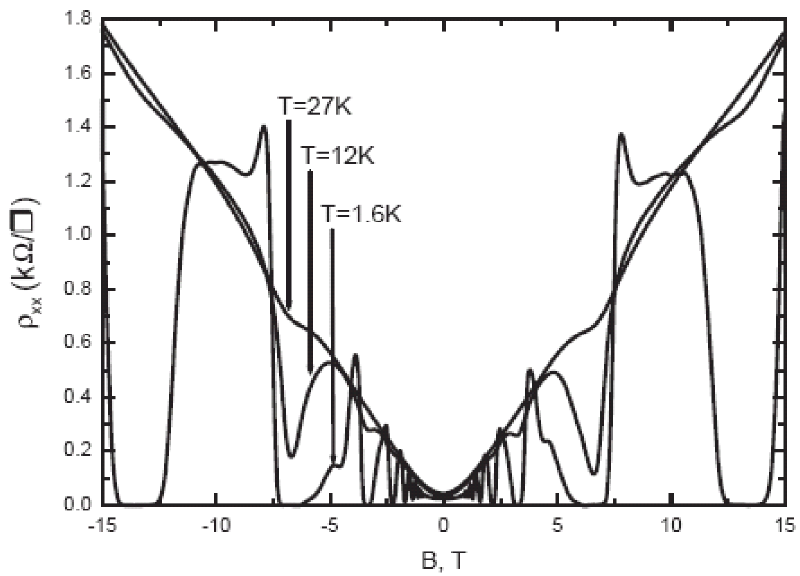


Fig. 2. Dissipative resistivity $\rho_{xx}(B)$ for the sample 188.

$$\rho_{xx}(B) \propto |B|$$

The behavior at high temperatures is fully accounted for by classical dynamics; at low temperatures the Quantum Hall effect is observed.

Resistance oscillations due to a periodic grating I

Instead of (or, in addition to) a long-range-correlated disorder impose a weak periodic grating $V(r) = V_0 \cos(2\pi y/a)$; observed resistance oscillations periodic in $1/B$ field, known as Weiss oscillations, with period determined by the condition

$$r_c = v_F/\omega_c = 2a/n, \quad n = 1, 2, 3...$$

(D. Weiss, K. v. Klitzing, K. Ploog, and G. Weimann, Europhys. Lett. 8, 179 (1989), where an optical grating was used).

Theory: cyclotron orbit drift enhanced or suppressed when the orbit radius is in or our of resonance.

Theory of Weiss oscillations

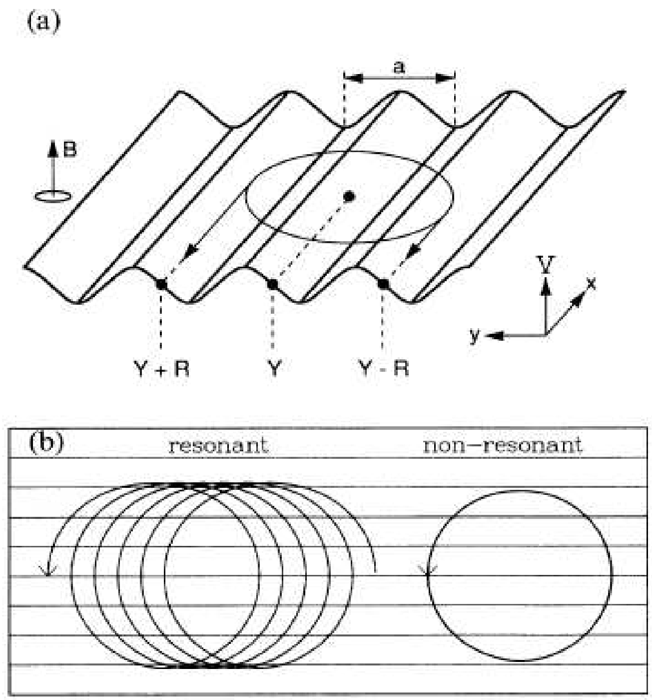


FIG. 1. (a) Potential grating with a cyclotron orbit superimposed. When the electron is close to the two extremal points $Y \pm R$ the guiding center at Y acquires an $\mathbf{E}\times\mathbf{B}$ drift in the direction of the arrows. (The drift along nonextremal parts of the orbit averages out, approximately.) A resonance occurs if the drift at one extremal point reinforces the drift at the other, as shown. (b) Numerically calculated trajectories for a sinusoidal potential (ϵ=0.015). The horizontal lines are equipotentials at integer y/a. On resonance ($2R/a$=6.25) the guiding-center drift is maximal; off resonance ($2R/a$=5.75) the drift is negligible.

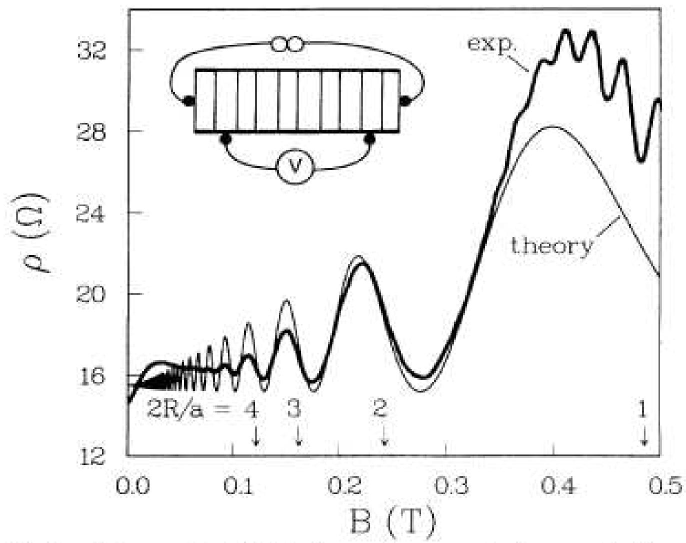
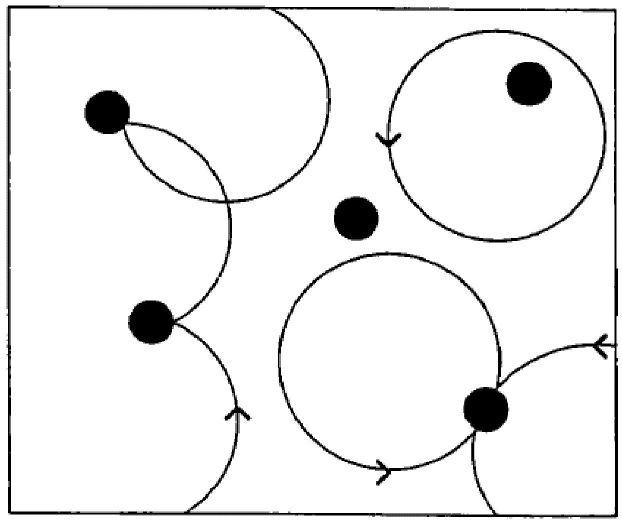


FIG. 2. Magnetic field dependence of the resistivity ρ_{yy} for current flowing perpendicular to the potential grating (see inset). The theoretical curve is from Eq. (6); the experimental curve from Ref. 1. Note the phase shift of the oscillations, as indicated by the arrows at integer $2R/a$. For $B \gtrsim 0.4$ T the experimental data show the onset of the Shubnikov–de Haas oscillations.

Taken from: Beenakker, PRL 62, 2020 (1989)

Magnetotransport of a Lorentz gas

Based on: A. V. Bobylev, F. A. Maa, A. Hansen, and E. H. Hauge, Phys. Rev. Lett. 75: 197 (1995).



Typical paths of the moving "electron" (charged particle) in a 2D Lorentz model a perpendicular magnetic field.

Classical dynamics of a charged particle in the presence of hard disks of density n and radius a. In the absence of magnetic field, the mean free path is $\ell = \Sigma/n = 2a/n$. Transport time $\tau = \ell/v_F$.

Applications: strong scatterers in a 2DEG (e.g. antidots)

The canonical Boltzmann kinetic equation:

spatially homogeneous case, with a one-particle distribution $f(\phi, t)$ depending on time t and on the direction $\phi = \angle(\mathbf{v}, \hat{x})$ of the velocity only. (The speed $v = |\mathbf{v}|$ is a constant of the motion here.) The BE reads

$$\left(\frac{\partial}{\partial t} + \omega \frac{\partial}{\partial \phi}\right) f(\phi, t) = \nu \int_{-\pi}^{\pi} d\psi\ g(\psi)[f(\phi - \psi, t) - f(\phi, t)] \equiv Bf(\phi, t)$$

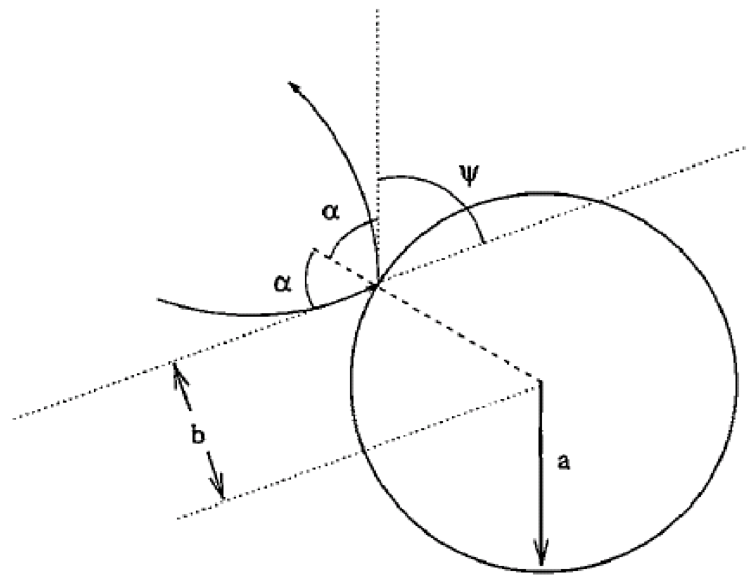
$$g(\psi) = \sigma(\psi)/\Sigma.$$

$\omega = e\mathcal{B}/m$ the cyclotron frequency $\quad \nu = nv\Sigma \quad \Sigma = \int d\psi\ \sigma(\psi) = 2a$

Here n is the density of scatterers (hard disks), a is the disk radius, $\sigma(\psi)$ is the differential scattering cross-section, $\Sigma = 2a$ is total cross-section.

In this form, B.k.e. is identical to what we studied before. Thus expect one can zero magnetoresistance and $\rho_{xy} = n/Be$.

Scattering on a hard disc



Scattering off a hard disk of radius a in a magnetic field. The impact parameter is b and the scattering angle is ψ, with $b = a \sin\alpha$ and $\alpha = (\pi - \psi)/2$.

the differential cross section

$$\sigma(\psi) = \left|\frac{db}{d\psi}\right| = \left|\frac{d}{d\psi}\left(a \sin\frac{\pi-\psi}{2}\right)\right| = \frac{a}{2}\sin\left|\frac{\psi}{2}\right|$$

in dimensionless version, $g(\psi) = \frac{1}{4}\sin|\psi/2|$, $\Sigma = 2a$.

The limit of small radius, high density

So-called Grad limit: $\tau = nv\Sigma$ constant, while $n \to \infty$, $a \to 0$ (for hard discs $\Sigma = 2a$).

> Note that in the Grad limit, where $a/R \to 0$, *two* important simplifications arise: (i) As pointed out above, the differential cross section becomes independent of the magnetic field. (ii) On the length scale set by the size of the scatterer, the cyclotron orbits degenerate into straight lines. This implies that the accumulated scattering angle after s successive encounters with the scatterer equals $s\psi$, where ψ is the scattering angle of the first collision.

Each scatterer is easy to miss, but if it is encountered once there's a high chance of comeback. Features of the dynamics in this limit:

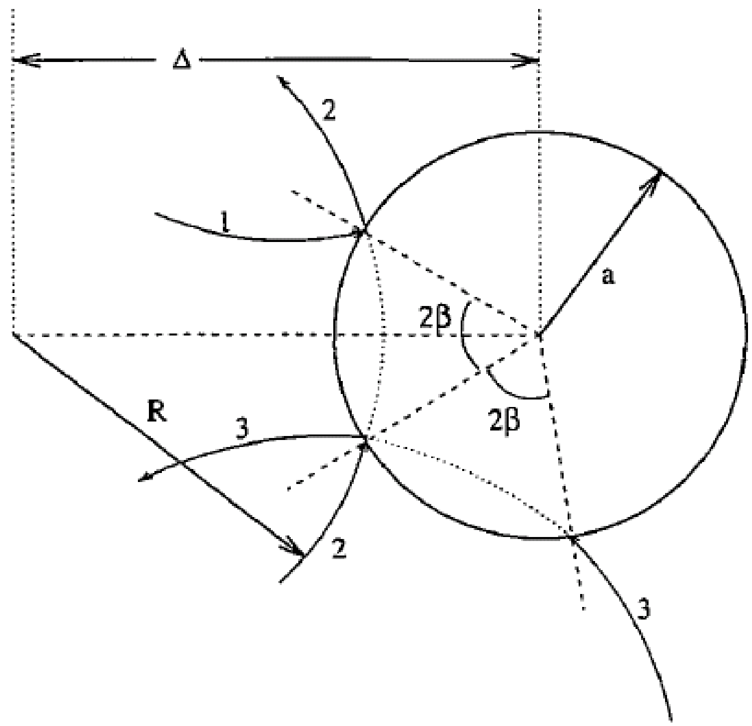
(i) An electron either does not collide with any scatterer or it collides (in the course of time) with infinitely many different ones. Exceptions are "measure zero;"

(ii) An electron can only recollide with a given scatterer only if no other scatterer has been hit in the mean time;

(iii) The total scattering angle with the same scatterer after s collisions is $s\psi$, where ψ is the scattering angle of the initial collision with this scatterer.

Why $s\psi$?

(iii) The total scattering angle with the same scatterer after s collisions is $s\psi$, where ψ is the scattering angle of the initial collision with this scatterer.



Successive collisions with a hard-disk scatterer of radius a. Subsequent cyclotron orbits (all with radius R) are numbered 1, 2, and 3. The initial collision makes the electron switch form cyclotron orbit 1 to cyclotron orbit 2, the second from orbit 2 to orbit 3, etc. The distance between the center of a cyclotron orbit and the center of the hard-disk scatterer in Δ. The angle separating two subsequent collision points on the periphery of the disk is 2β.

The limit of small radius, high density

GBE takes into account multiple encounters with each scatterer:

$$\left(\frac{\partial}{\partial t}+\omega\frac{\partial}{\partial\phi}\right)f^{\mathrm{G}}(\phi,t)=\sum_{s=0}^{[t/T]}P_0^s v\int_{-\pi}^{\pi}d\psi\; g(\psi)[f^{\mathrm{G}}(\phi-(s+1)\psi,t-sT)-f^{\mathrm{G}}(\phi-s\psi,t-sT)] \qquad (4)$$

Here $[t/T]$ is the number of cyclotron periods completed at time t. The generalized Boltzmann equation (4) is *exact* in the Grad limit.

The probability that a cyclotron orbit is free of scattering centers

$$P_0=\exp(-A_0 n)=\exp(-2\pi R\cdot 2na)\equiv\exp(-2\pi R/\Lambda)\quad \Lambda=1/(2na).$$

SOLUTION OF THE INITIAL VALUE PROBLEM

In order to solve the initial value problem posed by the GBE, we first introduce Fourier transforms in angles and Laplace transforms in time,

$$F_m(p) = \int_0^\infty dt\, e^{-pt} f_m(t) = \int_0^\infty dt\, e^{-pt} \int_{-\pi}^{\pi} d\phi\, e^{im\phi} f(\phi, t) \tag{5}$$

The Fourier transform of the right-hand side of the GBE reads, after a change in the order of integrations,

$$\nu \sum_{s=0}^{[t/T]} P_0^s \int_{-\pi}^{\pi} d\psi\; g(\psi)\, e^{ims\psi} (e^{im\psi} - 1)\; f_m^{\mathrm{G}}(t - sT) \tag{6}$$

With $P_0 = e^{-\nu T}$, the Fourier–Laplace transform of the RHS of the GBE reads, after a change in the order of summation over s with integrations over time and scattering angle,

$$\nu \int_{-\pi}^{\pi} d\psi\; g(\psi)\, \frac{e^{im\psi}}{1 - e^{-(\nu + p)T + im\psi}}\, F_m^{\mathrm{G}}(p) \tag{7}$$

The general solution of the initial value problem posed by the GBE is therefore, in Fourier–Laplace form,

$$F_m^{\mathrm{G}}(p)=\frac{(1-e^{-(p+\nu)T})f_m(0)}{p-im\omega+\nu\int_{-\pi}^{\pi}d\psi\ g(\psi)(1-e^{im\psi})/(1-e^{-(p+\nu)T+im\psi})} \qquad (9)$$

This result is valid for *any* interaction of short range. In order to perform the integration over ψ, the differential cross section corresponding to this interaction must be specified.

The Einstein–Kubo formula gives, with the understanding that $\phi(0)=0$,

$$\begin{aligned} D_{xx}=D_{yy}&=\int_0^{\infty}dt\langle v_x(t)\,v_x(0)\rangle=\tfrac{1}{2}v^2\int_0^{\infty}dt\langle\cos\phi(t)\rangle \\ D_{yx}=-D_{xy}&=\int_0^{\infty}dt\langle v_y(t)\,v_x(0)\rangle=\tfrac{1}{2}v^2\int_0^{\infty}dt\langle\sin\phi(t)\rangle \end{aligned} \qquad (12)$$

It is convenient to introduce the complex diffusion constant $\mathscr{D}=D_1+iD_2$, with $D_1=D_{xx}$ and $D_2=D_{yx}$. Then the Kubo formula acquires the compact form

$$\mathscr{D}=\tfrac{1}{2}v^2\int_0^{\infty}\langle e^{i\phi(t)}\rangle_{\phi(0)=0}=\tfrac{1}{2}v^2F_1(0) \qquad (13)$$

From (11) and (9) the diffusion tensor follows as[4]

$$\mathscr{D}=\frac{1}{2}v^2\left[\frac{e^{-\nu T}}{-i\omega}+\frac{1-e^{-\nu T}}{\nu\int_{-\pi}^{\pi}d\psi\; g(\psi)(1-e^{i\psi})/(1-e^{-\nu T+i\psi})-i\omega}\right] \qquad (14)$$

This result is valid for any interaction of short range.

We now specialize to hard disks by using (2), or its dimensionless Fourier transform, $g_m=-(4m^2-1)^{-1}$. Proceeding via infinite series, one can perform the ψ-integration in (14) for this case. The result is

$$\mathscr{D}=\frac{1}{2}v^2\left[\frac{x^2}{-i\omega}+\frac{\tau_D(x)(1-x^2)}{1-i\omega\tau_D(x)}\right];\qquad x=e^{-\nu T/2}=e^{-\pi\nu/\omega}=e^{-\pi m v/(Ae\mathscr{B})}$$
$$\tau_D^{-1}(x)=\nu\left[1-\frac{1-x^2}{2x^2}\left(\frac{1-x^2}{2x}\ln\frac{1+x}{1-x}-1\right)\right] \qquad (15)$$

The diffusion time scale $\tau_D(x)$ depends only weakly on the magnetic field.[5] As $\mathscr{B}\to 0$, i.e., $x\to 0$, (15) yields $\tau_D^{-1}=(4/3)\,\nu$, in agreement with the result from the standard Boltzmann equation. In the other extreme, $\mathscr{B}\to\infty$, i.e., $x\to 1$, one finds $\tau_D^{-1}=\nu$. (One can understand this physically by noting that, in this limit, the magnetic field acts as an effective randomizer of velocity directions.)

Magnetic field enhances diffusion time: negative magnetoresistance. Observed for 2DEGs with arrays of antidots.