Coherent and collective phenomena in quantum transport 8.513

- Lectures T, TR 1-2:30
- Homework handed out in class, collected two weeks later – except holidays and HW#1
- Term paper: choose topic by the end of October, submit before or in the last lecture
- Instructor: Leonid (aka Leo) Levitov, office 6C-345, email: levitov@mit.edu
- Course webpage: http://www.mit.edu/~levitov/8513

Quantum transport; Lecture I

- Localization vs. diffusion
- Anderson model, localization transition;
- Quantum-coherent effects in diffusive conductors: weak localization, negative magnetoresistance, Aharonov-Bohm effect ($2\Phi_0$ and Φ_0)
- Transport as a scattering problem: conductance = transmission; Landauer formula

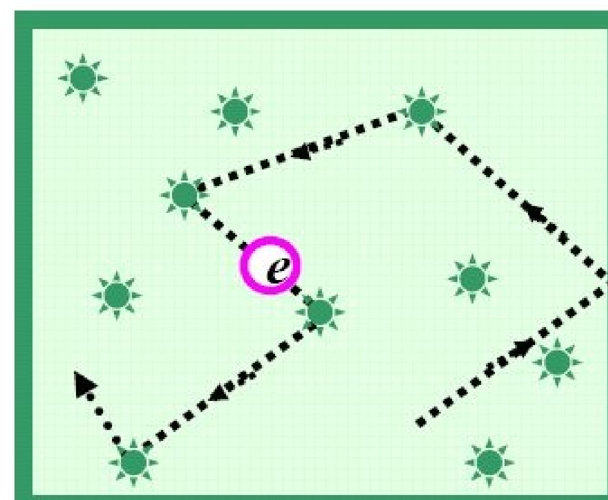
50 years of Anderson Localization

PHYSICAL REVIEW VOLUME 109, NUMBER 5 MARCH 1, 1958

Absence of Diffusion in Certain Random Lattices

P. W. ANDERSON
Bell Telephone Laboratories, Murray Hill, New Jersey
(Received October 10, 1957)

This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.



from lectures by Boris Altshuler

Philip W. Anderson

The Nobel Prize in Physics 1977

Nobel Lecture

Nobel Lecture, December 8, 1977

Local Moments and Localized States

I was cited for work both. in the field of magnetism and in that of disordered systems, and I would like to describe here one development in each held which was specifically mentioned in that citation. The two theories I will discuss differed sharply in some ways. The theory of local moments in metals was, in a sense, easy: it was the condensation into a simple mathematical model of ideas which. were very much in the air at the time, and it had rapid and permanent acceptance because of its timeliness and its relative simplicity. What mathematical difficulty it contained has been almost fully- cleared up within the past few years.

Localization was a different matter: very few believed it at the time, and even fewer saw its importance; among those who failed to fully understand it at first was certainly its author. It has yet to receive adequate mathematical treatment, and one has to resort to the indignity of numerical simulations to settle even the simplest questions about it .

Einstein Relation (1905)

$$\sigma = e^2 D\nu \qquad \nu \equiv \frac{dn}{d\mu}$$

Conductivity

Diffusion Constant

Density of states

No diffusion – no conductivity

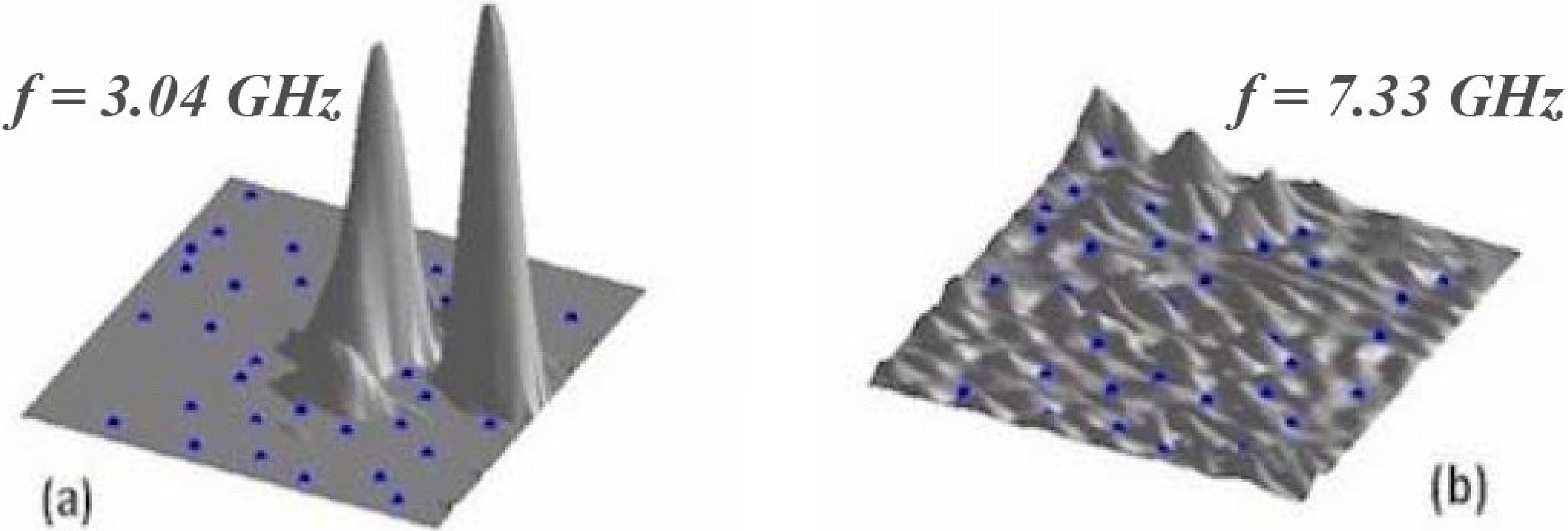
Localized states – insulator
Extended states - metal

Metal – insulator transition

VOLUME 85, NUMBER 11 PHYSICAL REVIEW LETTERS 11 SEPTEMBER 2000

Correlations due to Localization in Quantum Eigenfunctions of Disordered Microwave Cavities

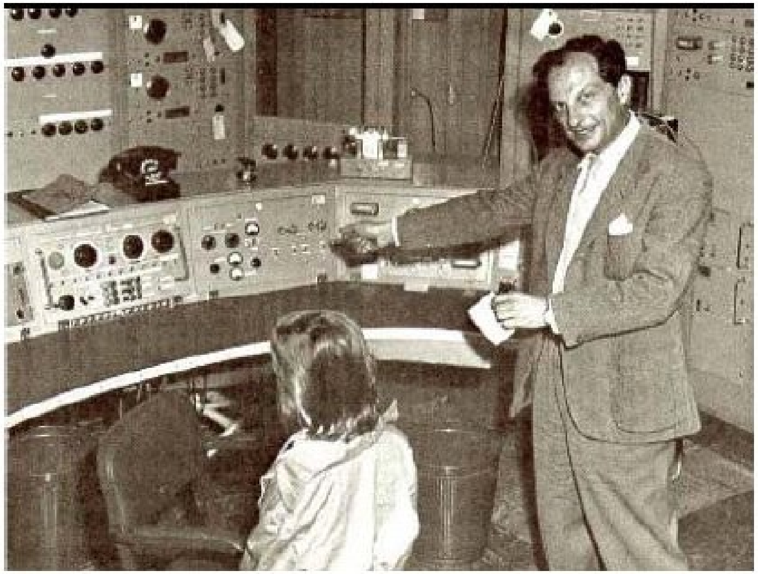
Prabhakar Pradhan and S. Sridhar
Department of Physics, Northeastern University, Boston, Massachusetts 02115
(Received 28 February 2000)



Anderson Insulator ***Anderson Metal***

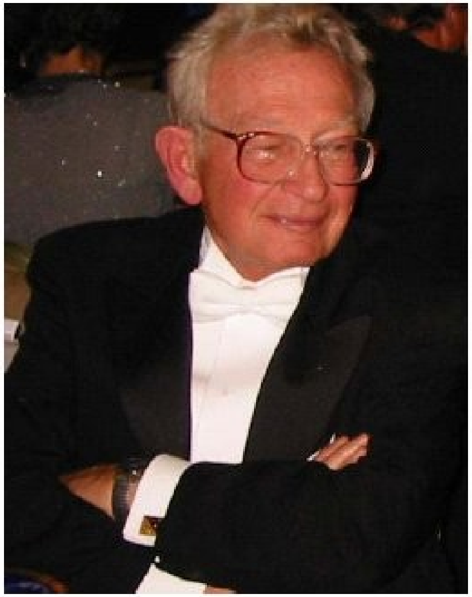
Fermi Pasta Ulam 1955

Q: Will a nonlinear system (system of interacting particles) completely isolated from the outside world evolve to a microcanonical distribution (reach equipartition)?



Anderson 1958

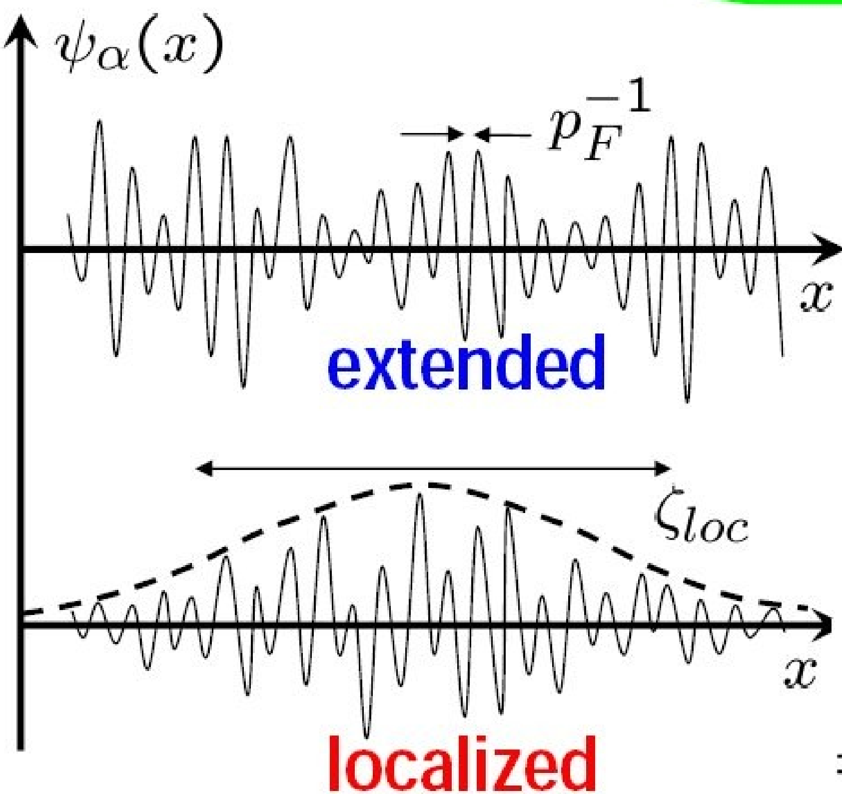
Q: Will a density fluctuation (a wave packet) in a system of quantum particles in the presence of disorder dissolve in the diffusive way?

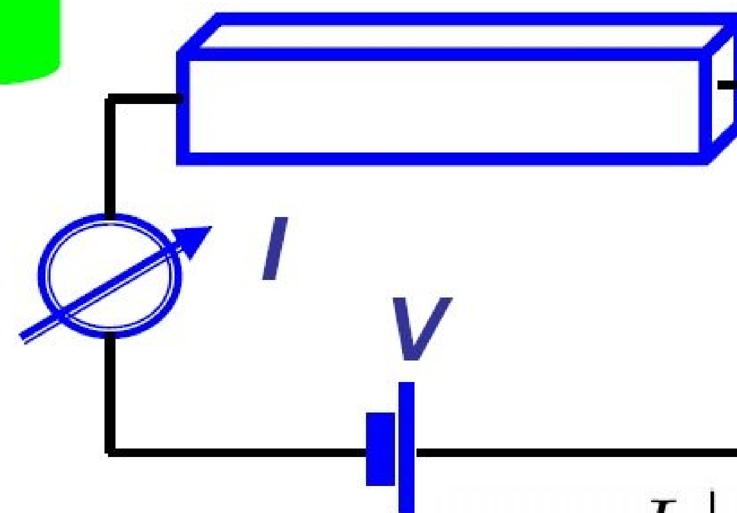


Localization of single-electron wave-functions:

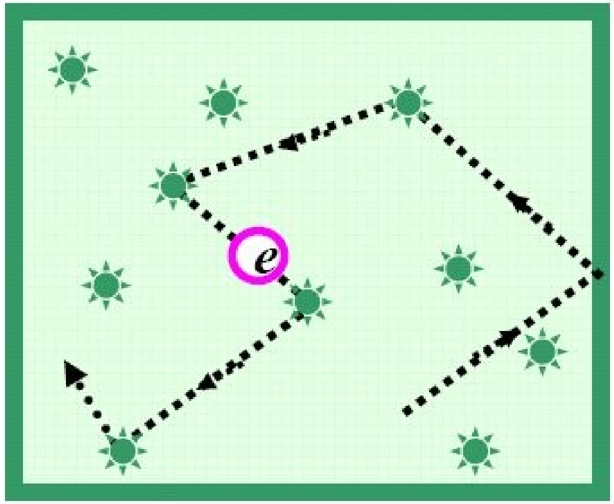
$$\left[-\frac{\boldsymbol{\nabla}^2}{2m}+U(\boldsymbol{r})-\epsilon_F\right]\psi_\alpha(\boldsymbol{r})=\xi_\alpha\psi_\alpha(\boldsymbol{r})$$

Disorder





Conductance $$G=\left.\frac{I}{V}\right|_{V\to 0}=\begin{cases}\sigma\frac{L_xL_y}{L_z}; & \text{extended}\\ \propto\exp\left(-L_z/\zeta_{loc}\right); & \text{localized}\end{cases}$$



Scattering centers, e.g., impurities

Models of disorder:

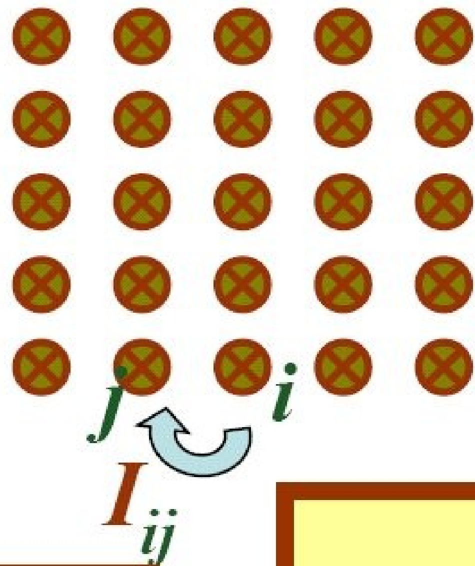
Randomly located impurities

White noise potential

Lattice models

- Anderson model
- Lifshits model

Anderson Model



- *Lattice - tight binding model*
- *Onsite energies* ε_i *- random*
- *Hopping matrix elements* I_{ij}

$$-W < \varepsilon_i < W$$

uniformly distributed

$$I_{ij} = \begin{cases} I & i \text{ and } j \text{ are nearest neighbors} \\ 0 & \text{otherwise} \end{cases}$$

Anderson Transition

$$\frac{I_c}{W} \simeq \left(\frac{1}{2d}\right)\left(\frac{1}{\ln d}\right)$$

$$I < I_c$$

Insulator

All eigenstates are localized

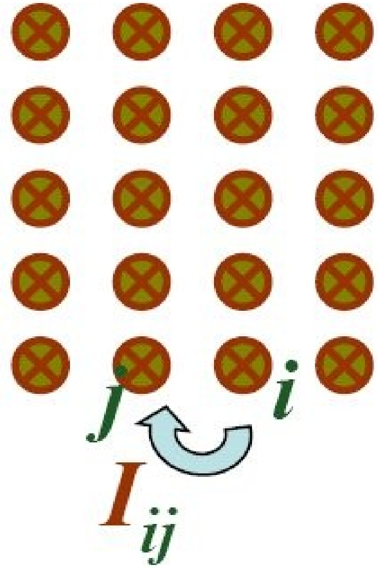
Localization length ξ

$$I > I_c$$

Metal

There appear states extended all over the whole system

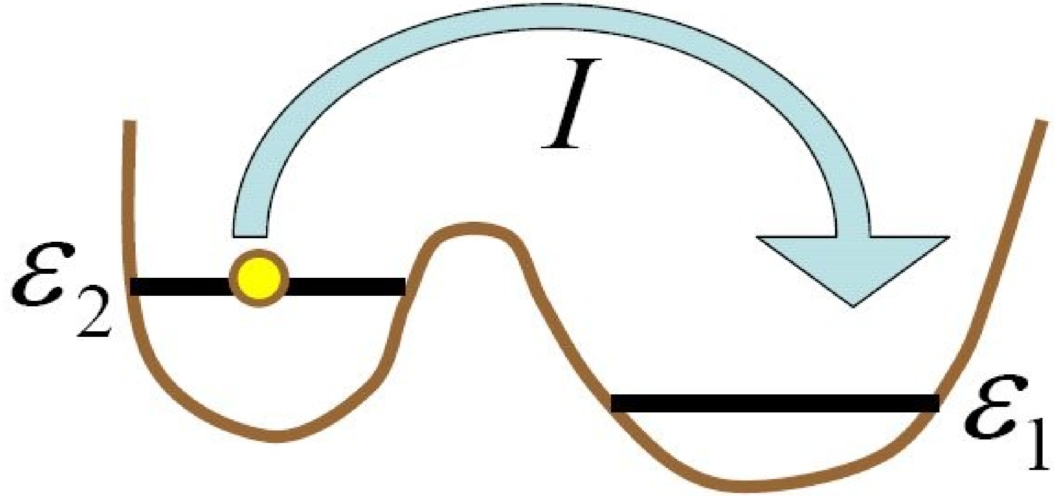
Q: Why arbitrary weak hopping I is not sufficient for the existence of the diffusion?



Einstein (1905): Marcovian (no memory) process → diffusion

Quantum mechanics is not marcovian
There is memory in quantum propagation!

Why?



Hamiltonian

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix} \xrightarrow{\text{diagonalize}} \hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$\boxed{E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2}}$$

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix} \xrightarrow{\text{diagonalize}} \hat{H} = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2} \approx \begin{matrix} \varepsilon_2 - \varepsilon_1 & \varepsilon_2 - \varepsilon_1 >> I \\ I & \varepsilon_2 - \varepsilon_1 << I \end{matrix}$$

$\Downarrow$

von Neumann & Wigner “noncrossing rule”

Level repulsion

What about the eigenfunctions ?

$$\hat{H} = \begin{pmatrix} \varepsilon_1 & I \\ I & \varepsilon_2 \end{pmatrix}$$

$$E_2 - E_1 = \sqrt{(\varepsilon_2 - \varepsilon_1)^2 + I^2} \approx \begin{matrix} \varepsilon_2 - \varepsilon_1 & \varepsilon_2 - \varepsilon_1 >> I \\ I & \varepsilon_2 - \varepsilon_1 << I \end{matrix}$$

What about the eigenfunctions ?

$$\varphi_1 \varepsilon_1; \varphi_2 \varepsilon_2 \Leftarrow \psi_1, E_1; \psi_2, E_2$$

$$\varepsilon_2 - \varepsilon_1 >> I$$

$$\psi_{1,2} = \varphi_{1,2} + O\left(\frac{I}{\varepsilon_2 - \varepsilon_1}\right)\varphi_{2,1}$$

Off-resonance

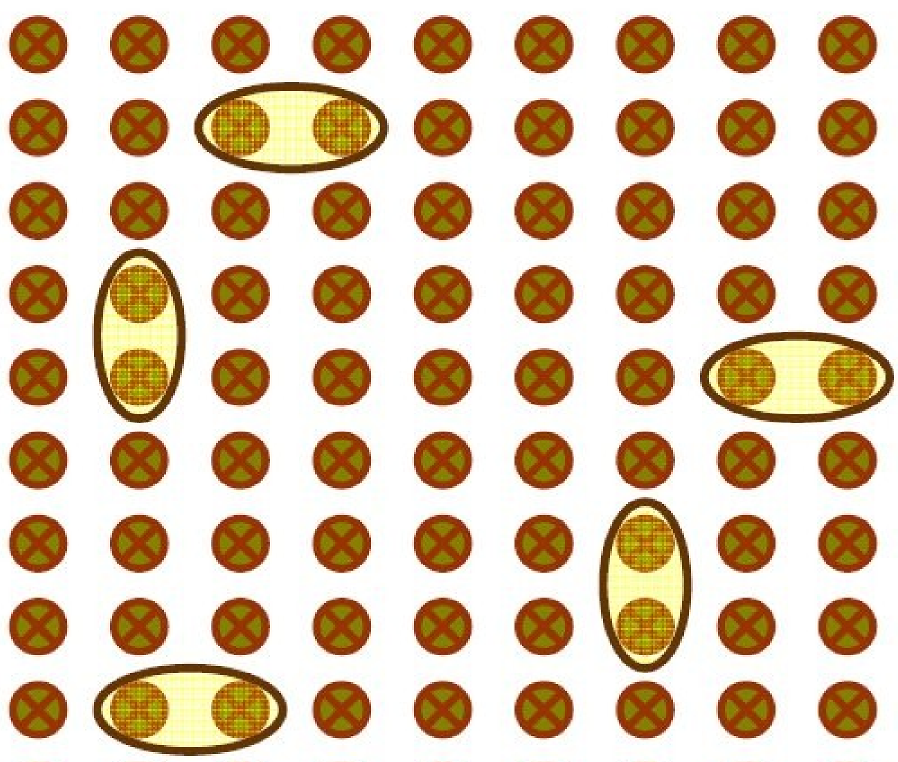
Eigenfunctions are close to the original on-site wave functions

$$\varepsilon_2 - \varepsilon_1 << I$$

$$\psi_{1,2} \approx \varphi_{1,2} \pm \varphi_{2,1}$$

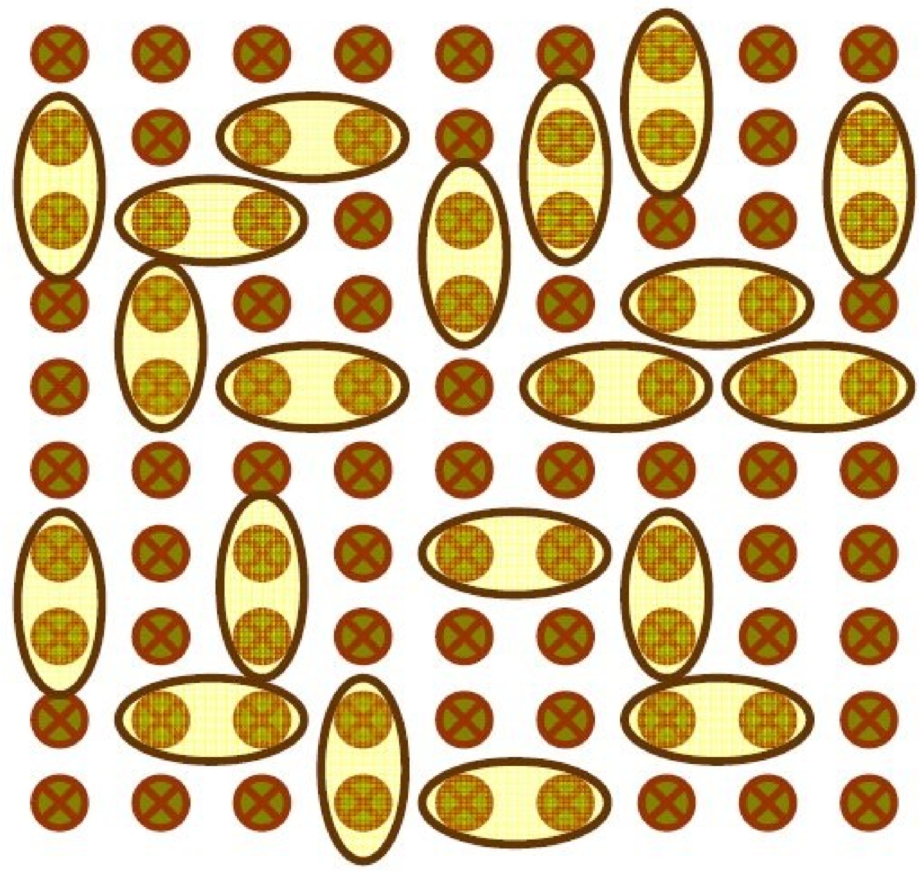
Resonance

In both eigenstates the probability is equally shared between the sites



Anderson insulator

Few isolated resonances



Anderson metal

There are many resonances and they overlap

Simplest example: Anderson Model Cayley tree:

J. Phys. C: Solid State Phys., Vol. 6, 1973. Printed in Great Britain. © 1973

A selfconsistent theory of localization

R Abou-Chacra†, P W Anderson‡§ and D J Thouless†

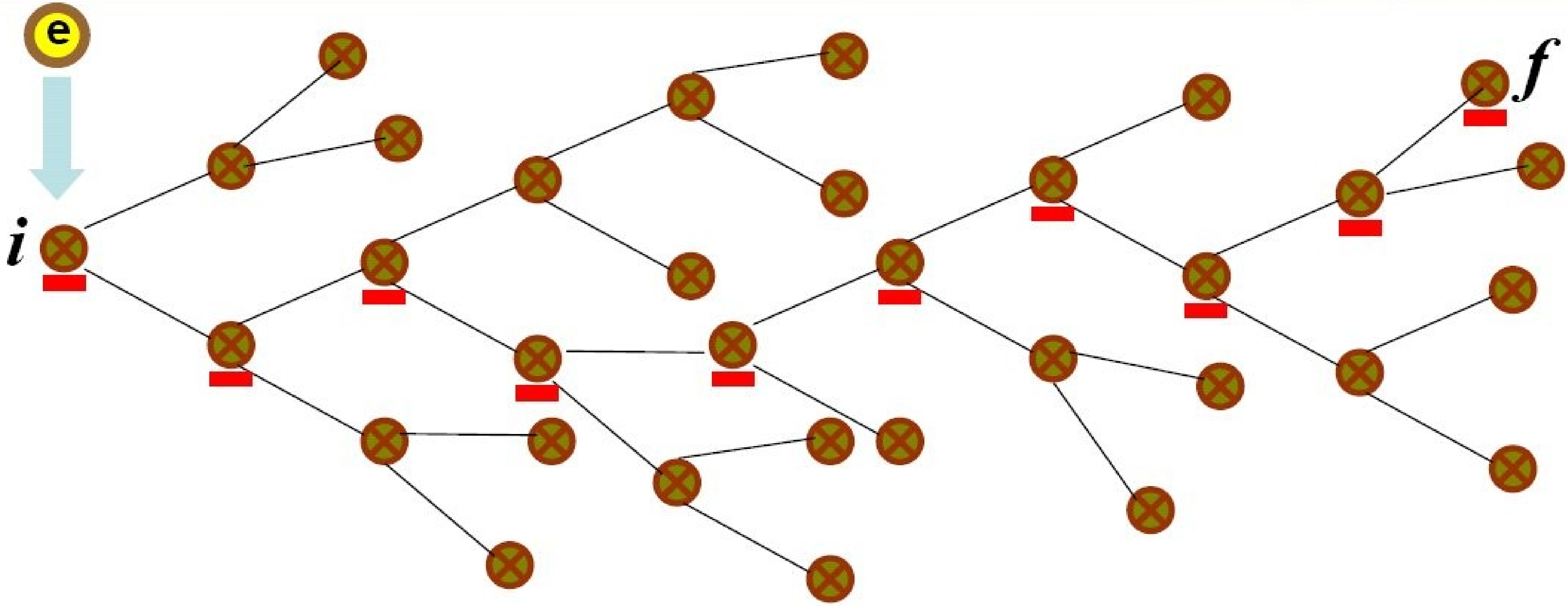
† Department of Mathematical Physics, University of Birmingham, Birmingham, B15 2TT

‡ Cavendish Laboratory, Cambridge, England and Bell Laboratories, Murray Hill, New Jersey, 07974, USA

Received 12 January 1973

Abstract. A new basis has been found for the theory of localization of electrons in disordered systems. The method is based on a selfconsistent solution of the equation for the self energy in second order perturbation theory, whose solution may be purely real almost everywhere (localized states) or complex everywhere (nonlocalized states). The equations used are exact for a Bethe lattice. The selfconsistency condition gives a nonlinear integral equation in two variables for the probability distribution of the real and imaginary parts of the self energy. A simple approximation for the stability limit of localized states gives Anderson's 'upper limit approximation'. Exact solution of the stability problem in a special case gives results very close to Anderson's best estimate. A general and simple formula for the stability limit is derived; this formula should be valid for smooth distribution of site energies away from the band edge. Results of Monte Carlo calculations of the selfconsistency problem are described which confirm and go beyond the analytical results. The relation of this theory to the old Anderson theory is examined, and it is concluded that the present theory is similar but better.

Simplest example: Anderson Model Cayley tree:



Parameters: I, W and branching number K (here $K=2$)

Crucial simplification: no loops

The probability amplitude to find the particle at a distance n is proportional to

$$A(n) \propto I^n \prod_{j=1}^{n} \frac{1}{\varepsilon - \varepsilon_j} \approx I^n \left(\frac{K}{W} \right)^n$$

The probability amplitude to find the particle at a distance ***n*** is proportional to

$$A(n) \propto I^n \prod_{j=1}^{n} \frac{1}{\varepsilon - \varepsilon_j} \approx I^n \left(\frac{K}{W} \right)^n$$

At each step among ***K*** site we can choose the one, which energy is the closest to $\boldsymbol{\varepsilon}$, i.e., $|\varepsilon - \varepsilon_j| \approx W/K$

K>1: Competition between exponentially small amplitude of each path and exponentially large number of paths.

Conclusion: for $I < I_c$, where $I_c \approx W/K$ the system is an insulator, because $A(n \to \infty) \to 0$ In the opposite case – metal

More precisely $I_c \approx W/(K \log K)$

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More precisely $I_c \approx W/(K \log K)$

$I > W/K$ Typically there is a resonance at every step

$W/(K \log K) < I < W/K$ The particle can travel infinitely far through the resonances of sites, which are not nearest neighbors

$I > W$ Typically each pair of nearest neighbors is at resonance

Quantum effects in diffusive transport

from lectures by Ben Simons

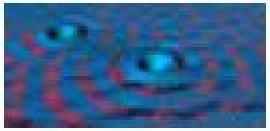
Weak localization

Phase Coherence Phenomena in Normal Conductors

How does disorder influence transport?

▷ Classically, from Drude theory

$$\boxed{\sigma = \frac{ne^2\tau}{m}} \qquad 1/\tau \quad \text{scattering rate}$$



▷ Quantum mechanically? Naively...

(i) for $\ell \equiv v_F\tau \gg \lambda_F$, interference is unimportant & can add intensities $\rightsquigarrow$ Drude!

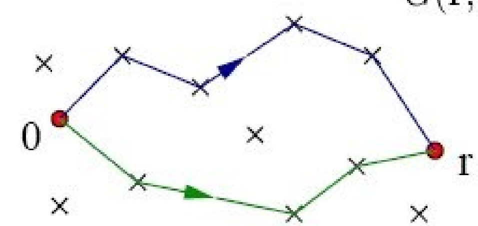
(ii) for $\ell \sim \lambda_F$ conductivity vanishes: metal-insulator transition

...However, mechanisms of quantum interference impact even on metallic phase

Feynman Trajectories

▷ Transfer probability amplitude:

$$G(\mathbf{r}, 0; t) = \langle \mathbf{r}| e^{-i\hat{H}t/\hbar}|0\rangle\theta(t)$$



$$\boxed{G(\mathbf{r}, 0; t) = \int D\mathbf{r}(t)\ \exp\left[\frac{i}{\hbar}S[\mathbf{r}(t)]\right] = \sum_i A_i\, e^{i\varphi_i}}$$

▷ Impurity average

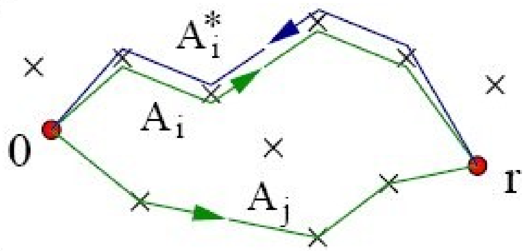
i.e. over realisations of random impurity potential

$$\langle G(\mathbf{r}, 0; t)\rangle_V = \left\langle \sum_i A_i\, e^{i\varphi_i}\right\rangle_V \sim e^{-t/2\tau}$$

i.e. random phase cancellation $\rightsquigarrow$ short-range correls.

Quantum diffusion

▷ Probability *density*:



$$G(\mathbf{r},0;t) = \sum_i A_i e^{i\varphi_i}$$

$$\overbrace{\left\langle |G(\mathbf{r},0;t)|^2 \right\rangle_V}^{\langle P(\mathbf{r},t)\rangle_V} = \overbrace{\left\langle \sum_i A_i^2 \right\rangle_V}^{\text{classical}} + \overbrace{\left\langle \sum_{i\neq j} A_i A_j \cos(\varphi_i - \varphi_j) \right\rangle_V}^{\text{quantum} \overset{\langle\cdots\rangle_V}{\to} 0}$$

▷ 'Classical contribution'

$\rightsquigarrow$ long-ranged diffusion, $D = v_F^2 \tau / d$

$$\boxed{\partial_t P(\mathbf{r},t) - D\partial^2 P(\mathbf{r},t) = \delta(\mathbf{r})\delta(t)}$$

& predicts classical Drude conductivity $\sigma = e^2 \nu D$

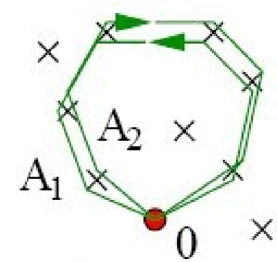
...but is this the whole story...?

Quantum Interference...

...leads to new phenomena which impact on *metallic* regime well before Anderson transition!

▷ Consider, e.g., *return* probability:

contrib. from time-reversed paths:



$$|A_1 e^{i\varphi_1} + A_2 e^{i\varphi_2}|^2 = \overbrace{A_1^2 + A_2^2}^{\text{classical}} + \overbrace{2A_1 A_2 \cos(\varphi_1 - \varphi_2)}^{\text{quantum}}$$

In *T-invariant* system, $A_1 = A_2$, $\varphi_1 = \varphi_2$

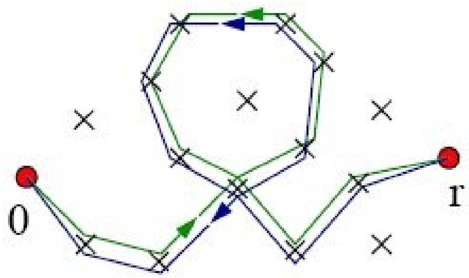
i.e. $\boxed{P_{\text{quantum}} = 2 \times P_{\text{classical}}}$

▷ Constructive interference $\rightsquigarrow$ tendency to localize

$\rightsquigarrow$ quantum correction to conductivity...

Suppression of conductance

Weak Localisation[1]



$$\frac{\delta\sigma}{\sigma} = \frac{\delta P}{P} \sim -\int dP_{\text{return}} \simeq -v_F \lambda_F^{d-1} \int_{\tau}^{L^2/D} \frac{dt}{(Dt)^{d/2}}$$

$\rightsquigarrow$ singular correct$\underline{\text{ns.}}$ in low dim.

$$\boxed{\delta\sigma_{d=2} = -\frac{e^2}{\pi^2\hbar} \ln\left(\frac{L}{\ell}\right)}$$

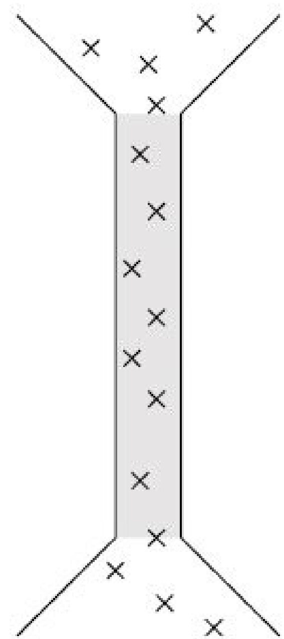
...accumulation of WL correct$\underline{\text{ns.}}$: scaling[2]

$\rightsquigarrow$ Anderson localisation

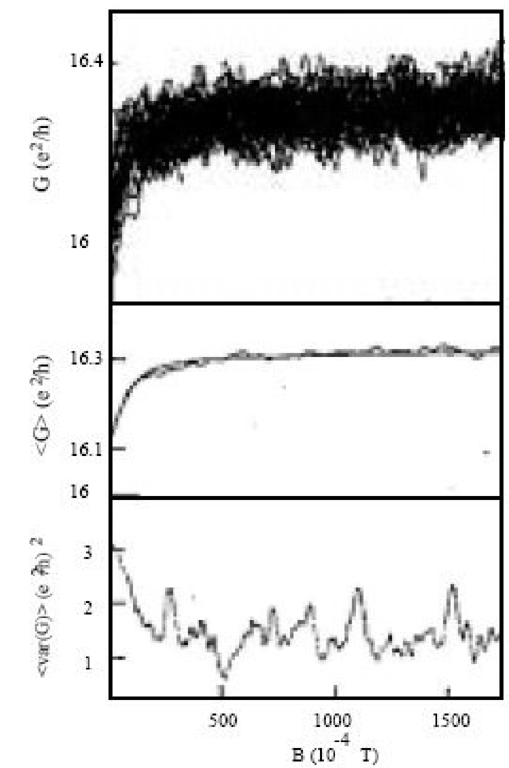
[1] Gor'kov, Larkin & Khmel'nitskii '79
[2] Abrahams, Anderson, Ramakrishnan & Liciardello '79

Experimental Signatures: Negative Magnetoresistance

- Quantum Wires[1]



SiGaAs Wire

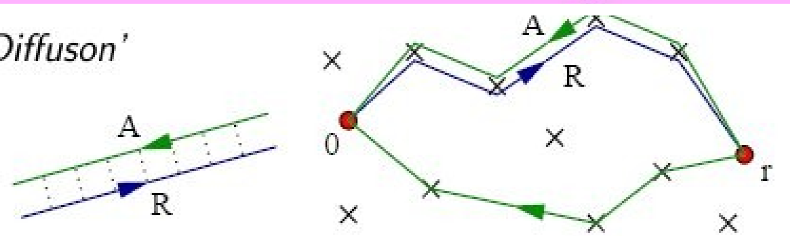


[1] Mailly and Sanquer '90

Long-range quantum coherence

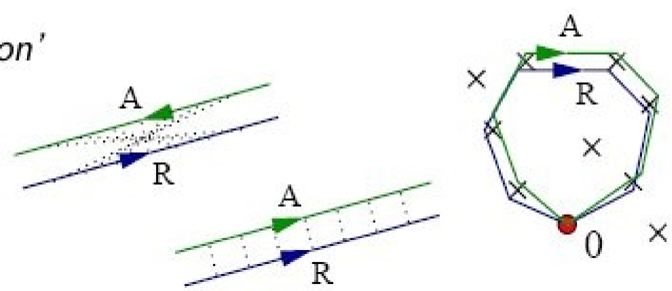
▷ Quantum interference effects mediated by two long-ranged modes of density relaxation...

- *'Diffuson'*



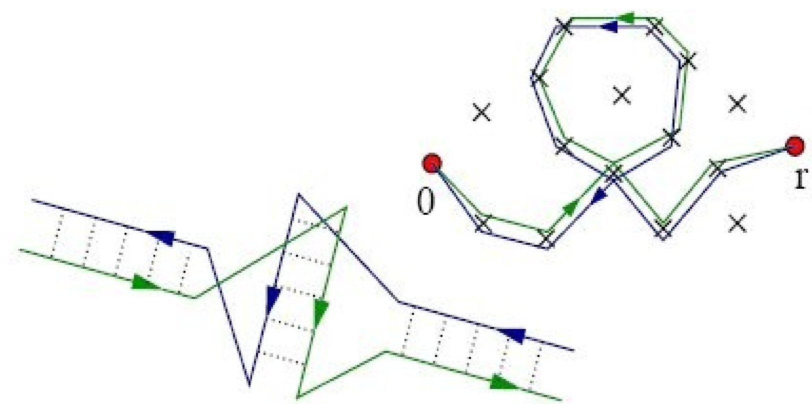
$$\langle G^R(0,\mathbf{r};t)G^A(\mathbf{r},0;t)\rangle_V \stackrel{\mathrm{FT}}{=} \frac{2\pi\nu}{D\mathbf{q}^2+i\omega}$$

- *'Cooperon'*



$$\langle G^R(0,\mathbf{r};t)\overbrace{G^A(0,\mathbf{r};t)}^{[G^A]^T}\rangle_V \stackrel{\mathrm{FT}}{=} \frac{2\pi\nu}{D\mathbf{q}^2+i\omega}$$

...together, diffusion modes form basic elements of diagrammatic perturbation theory



⇝ characteristic weak localization phenomena

▷ But magnetoresistance is not the only physical manifestation of quantum interference...

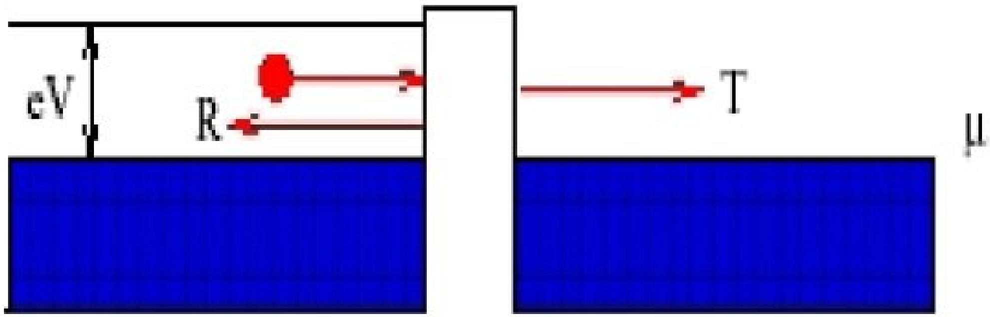
...effects recorded in fluct$^{\underline{\mathrm{n.}}}$ phenomena...

Scattering approach

from lectures by Markus Buttiker

Conductance from transmission

Heuristic discussion



Fermi energy left contact $\mu + eV$

Fermi energy right contact μ ,

applied voltage eV ,

transmission probability T ,

reflection probability R ,

incident current $I_{in} = ev_F \Delta\rho$

density $\Delta\rho = (d\rho/dE)\, eV$

density of states $d\rho/dE = (d\rho/dk)\,(dk/dE) = (1/2\pi)\,(1/\hbar v_F)$

$\Rightarrow$ $I_{in} = (e/h)eV$ independent of material !!

$I = (e/h)TeV \Rightarrow$

Landauer formula

$$\boxed{G = dI/dV = \frac{e^2}{h}T}$$

Drift and diffusion

$$j = \sigma E - eD dn/dx$$

$$E = -dU/dx$$

$$dn = \nu\, d\mu - e\,\nu\, dU$$

Electrochemical potential μ-eU

at constant μ

Einstein relation

$$j = 0 = -\sigma dU/dx + e^2\, D\nu\, (dU/dx) \quad \Rightarrow \quad \boxed{\sigma = e^2\, D\nu}$$

for space dependent μ

$$j = -e\,\nu\, D\, d\mu/dx \quad \Rightarrow \quad j = e\,\nu\, D\,(\mu_L - \mu_R)/L$$

$$j = (e^2\,\nu\, D\,/\,L)V \quad \longrightarrow \; \longleftarrow \quad I = \frac{e^2}{h} T\, V$$

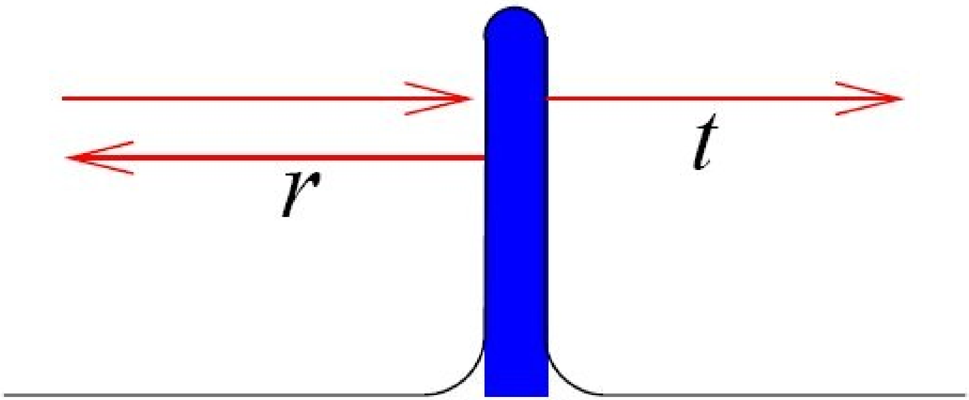
Scattering matrix

scattering state

$$|\Psi>_{inc} = e^{ikx}$$

$$|\Psi>_{ref} = re^{-ikx}$$

$$|\Psi>_{tra} = t\, e^{ikx}$$



scattering matrix

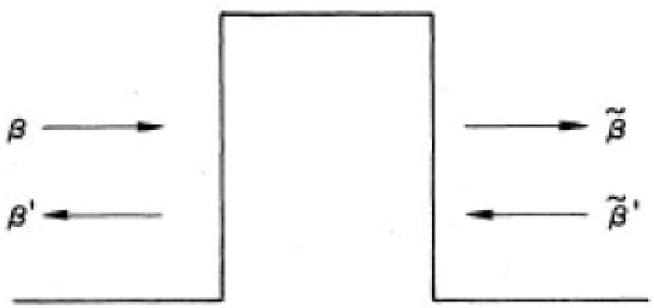
$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

current conservation $\Rightarrow$ S is a unitray matrix

In the absence of a magnetic field S is an orthogonal matrix

$$t' = t$$

Transfer matrix



$$\begin{pmatrix} \tilde{\beta} \\ \tilde{\beta}' \end{pmatrix} = \begin{pmatrix} 1/t^\star & -r^\star/t^\star \\ -r/t & 1/t \end{pmatrix} \begin{pmatrix} \beta \\ \beta' \end{pmatrix}$$

Transfer matrix is muliplicative $\Rightarrow$ arbitrary array of scatterers

One dimensional localization:

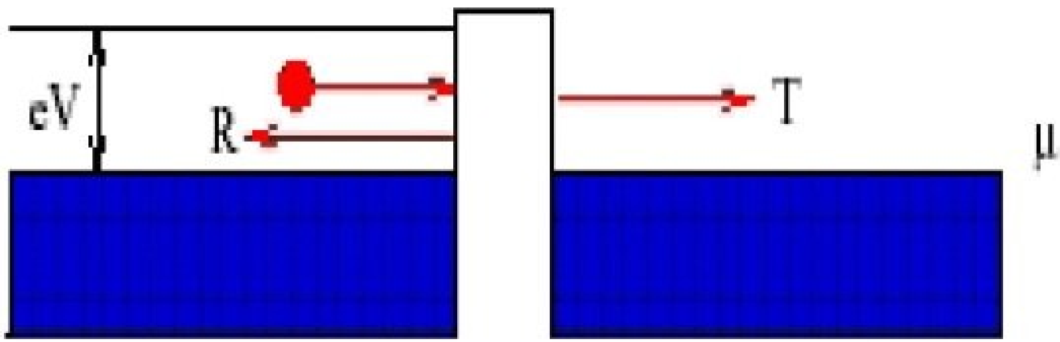
$$\langle G \rangle = \frac{e^2}{h} \exp(-L/\lambda)$$ localization length λ

but $\log G$ is normal distributed

characterize the sample through its distribution

$P(G)\,dG$ Discuss again later

Conductance from transmission



$$G = dI/dV = \frac{e^2}{h} T$$

$$\mathcal{R} = dV/dI = \frac{h}{e^2} \frac{1}{T}$$

conductance quantum

$$\frac{e^2}{h}$$

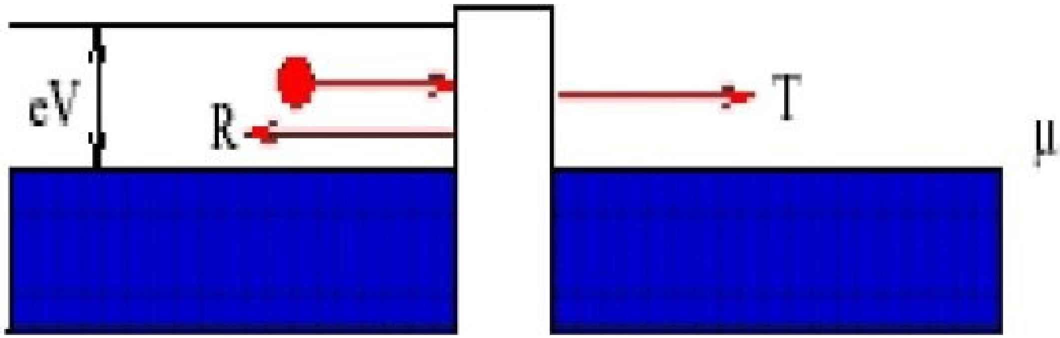
resistance quantum

$$\frac{h}{e^2} \approx 24\, kOhm$$

dissipation and irreversibility

$$W = I\,V = G\,V^2$$

Conductance from transmission



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conductance quantum

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resistance quantum

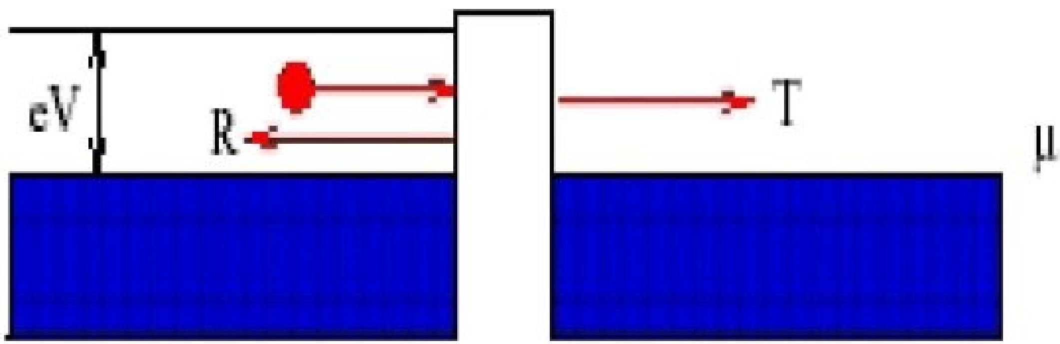
$$\frac{h}{e^2} \approx 24\, kOhm$$

dissipation and irreversibility

$$W = IV = GV^2$$

Dissipation for elastic scattering?!

Conductance from transmission



$$G = dI/dV = \frac{e^2}{h} T$$

conductance quantum

$$\frac{e^2}{h}$$

$$\mathcal{R} = dV/dI = \frac{h}{e^2} \frac{1}{T}$$

resistance quantum

$$\frac{h}{e^2} \approx 24\, kOhm$$

dissipation and irreversibility

$$W = IV = GV^2$$

Dissipation for elastic scattering?!
Energy is lost to the reservoirs.

General properties of S-matrix

Current conservation

Scattering matrix is a unitary matrix

$$s = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$

$$s^\dagger s = 1$$

$$r^\star r + t^\star t = 1 \quad \Rightarrow \quad R + T = 1 \,,$$

$$t'^\star r + r'^\star t = 0$$

$$r^\star t' + t^\star r' = 0$$

$$t'^\star t' + r'^\star r' = 1 \quad \Rightarrow \quad R' + T' = 1 \,,$$

$$s s^\dagger = 1$$

$$\Rightarrow \quad R' + T = 1$$

$$\Rightarrow \quad R + T' = 1$$

$$\Longrightarrow \quad T' = T \,, \quad R' = R$$

Magnetic field symmetry

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = s(B) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Time reversal: Hamiltonian is invariant when all momenta and B field are reversed

$$\begin{pmatrix} a_1^\star \\ a_2^\star \end{pmatrix} = s(-B) \begin{pmatrix} b_1^\star \\ b_2^\star \end{pmatrix} \Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = s^\star(-B) \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\Rightarrow s^\star(-B)s(B) = 1 \Rightarrow \ s^\dagger(B) = s^\star(-B) \ \Rightarrow$$

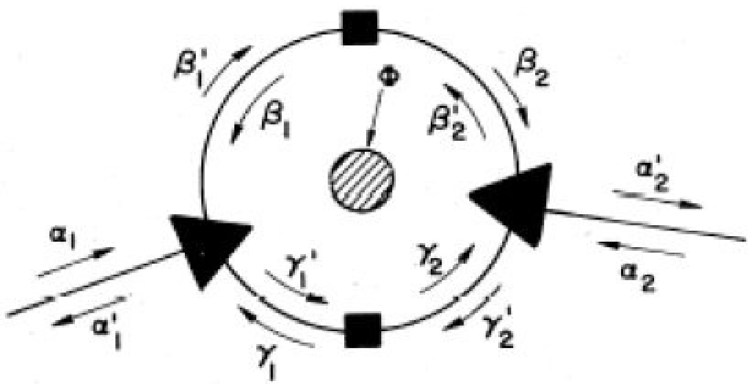
$$s^T(B) = s(-B)$$

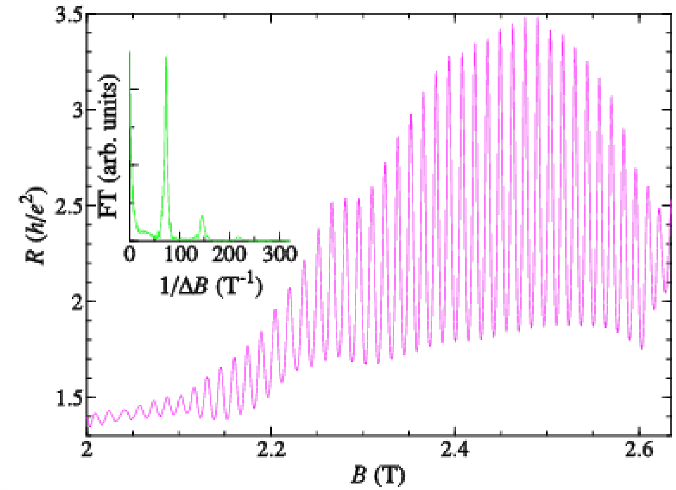
$$t'(B) = t(-B) \qquad \Rightarrow T'(B) = T(-B)$$

but $\quad T'(B) = T(B) \ \Rightarrow \ T(B) = T(-B)$

$G = dI/dV = \dfrac{e^2}{h}T$ is an even function of magnetic field

Aharonov-Bohm conductance oscillations





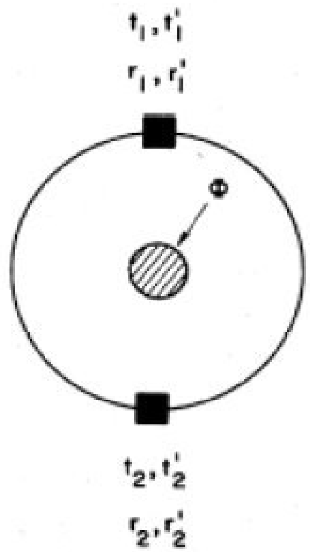
$$\chi_1 = \int_{upper} ds\mathbf{A} \;, \chi_2 = \int_{lower} ds\mathbf{A} \;, \quad \chi_1 - \chi_2 = 2\pi\Phi/\Phi_0,$$

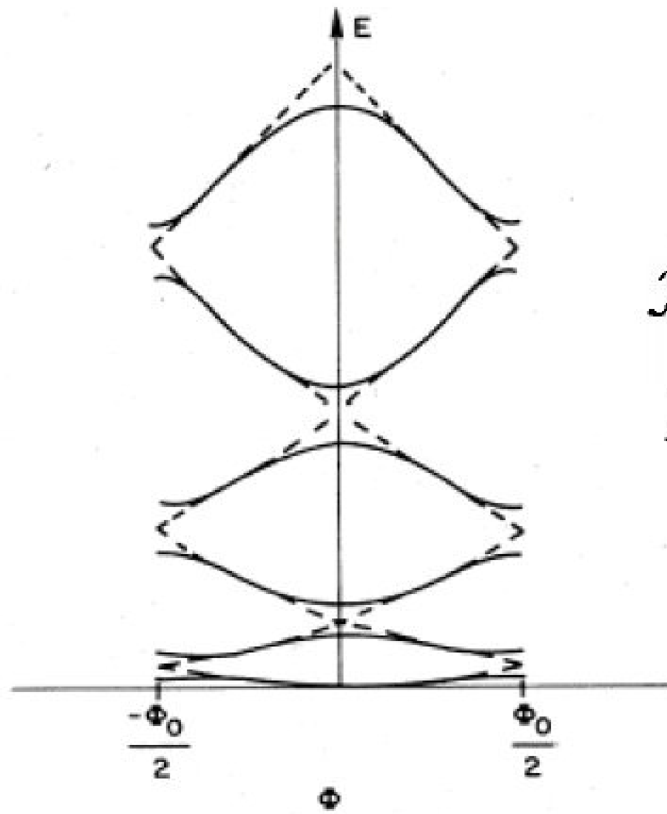
$$G(\Phi) = \frac{e^2}{h} T(\Phi)$$

$$G(\Phi) = \sum_n G_n \, cos(2\pi n\Phi/\Phi_0)$$

Buttiker, Imry, Azbel, Phys. Rev. A30, 1982 (1984)

Persistent current
(periodic boundary conditions)





$$\chi = \int_{circle} ds\, \mathbf{A} = 2\pi\Phi/\Phi_0 \ ,$$

$$\mathcal{F} = \sum_n E_n(\Phi) \ ,$$

$$I = -d\mathcal{F}/d\Phi$$

$$k = -(2\pi/L)(\Phi/\Phi_0) \ ,$$

$$I = -d\mathcal{F}/d\Phi = -\sum_n dE_n(\Phi)/d\Phi \ = -\sum_n (dE_n(k)/dk)(dk/d\Phi)$$

$$I = (e/L)\sum_n v_n(k)$$

General properties of the S-matrix

Unitarity

Current conservation

$$\mathcal{E}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + U\psi \quad \times \varphi^*$$

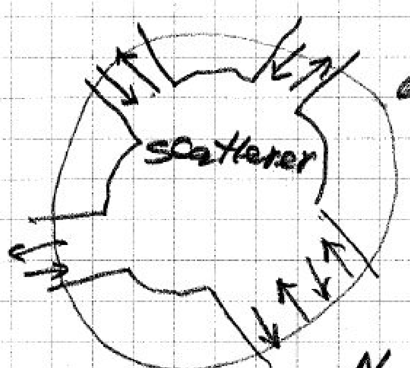
$$\mathcal{E}\varphi^* = -\frac{\hbar^2}{2m}\nabla^2\varphi^* + U\varphi^* \quad \times \psi$$

$$\Rightarrow \nabla\cdot j(\psi,\varphi) = 0$$

$$\vec{j}_{12} = -\frac{i\hbar}{2m}\varphi^*\vec{\nabla}\psi + c.c.$$

Integral form:

$\oint \vec{j}\cdot d\vec{a} = 0$ for any closed surf D



Focus on non-oscillatory terms (drop $\psi_{in}\psi_{out}$ etc.)

$$\oint\left(-\frac{i}{2}\varphi^*_{in}\vec{\nabla}\psi_{in} - \frac{i}{2}\varphi^*_{out}\vec{\nabla}\psi_{out} + c.c\right)d\vec{a} = 0$$

Decompose into N scatt. modes:

$$\sum_{j=1}^{N} \varphi^*_{j,in}\psi_{j,in} = \sum_{j=1}^{N} \varphi^*_{j,out}\psi_{j,out}$$

Substitute $\psi_{out} = S\psi_{in}$, $\varphi_{out} = S\varphi_{in}$

$\langle S\varphi_{in}|S\psi_{in}\rangle = \langle\varphi_{in}\psi_{in}\rangle$ (for any ψ, φ)

Hence $S^\dagger S = 1$, or $S^\dagger = S^{-1}$

Example 1: single channel, a 2x2 unitary matrix

Example 2: many channels, an NxN unitary matrix, eigenvalues of the form $S_j = \exp(2i\delta_j)$

δ_j scattering phases

Causality in EM and in QM

Cause-effect relation, linear response

$$\mathscr{D}(t) = \int_0^\infty \mathscr{K}(\tau)\,\mathscr{E}(t-\tau)\,d\tau;$$

$$\mathscr{E} = \int \mathscr{E}_\omega e^{-i\omega t}\,d\omega, \qquad \mathscr{D} = \int \mathscr{D}_\omega e^{-i\omega t}\,d\omega$$

$$\mathscr{D}_\omega = \mathscr{E}_\omega \int_0^\infty \mathscr{K}(\tau)\,e^{i\omega\tau}\,d\tau,$$

$$\varepsilon_\omega = \frac{\mathscr{D}_\omega}{\mathscr{E}_\omega} = \int_0^\infty \mathscr{K}(\tau)\,e^{i\omega\tau}\,d\tau.$$

Dielectric function ε(ω) analytic in the upper half-plane

$$\omega = \omega_0 + i\omega_1, \quad \omega_1 > 0,$$

b/c of exponentially decreasing $e^{-\omega_1\tau}$.

out-state in-state

$$B = SA$$

$A(t) = 0$ at $t < 0$, $B(t) = 0$ at $t < t_1$, $t_1 > 0$

$$A_\omega = \int_0^\infty A(t)\,e^{i\omega t}\,dt, \quad B_\omega = \int_{t_1}^\infty B(t)\,e^{i\omega t}\,dt.$$

$$B_\omega = SA_\omega$$

S(ω) could have singularities at the zeros of A(ω), but this is impossible b/c S is a property of the scattering potential, independent of the in-state

As a function of energy, S-matrix is analytic is the upper complex half-plane