Chapter 3

Particle in a Random Potential

In this chapter we shall introduce the diagrammatic impurity-averaging technique, which will be our basic tool for studying the physical properties of systems with quenched disorder. After introducing the concept of a random potential, we first study the average propagator order by order in perturbation theory, and finally the average density matrix. In the course of this we shall show how to partially sum perturbation expressions based on the topological structure of diagrams, and encounter the important concepts of self-energy and skeleton diagrams.

A metallic conductor exhibits at low temperatures a temperature-independent resistance, the value of which is called the residual resistance. The residual resistance is due to the deviation of the sample from that of an ideal crystal. A conductor always has imperfections: foreign atoms substituting for atoms of the crystal, vacancies due to missing atoms, dislocations in the crystal, grain boundaries, etc. These defects will scatter an electron in the conductor between the different current-carrying eigenstates of the ideal crystal Hamiltonian, and thus cause current degradation. The effect on the electron motion of such imperfections we can model as giving rise to a potential deviating from that of the ideal crystal. We shall call this potential the impurity potential, and view it as a result of *impurities*. We shall in the following assume static impurities characterized by an effective potential. This is in contrast to the case where a defect has several energy states available, between which it can make transitions either by thermal excitation or by quantum tunneling. We shall in chapter 11 discuss a case where the dynamics of the defects is of importance, viz. electron-phonon interaction in dirty metals.

3.1 Random Potential Model

Consider that in a given sample we have impurities located at some definite positions $\mathbf{r}_i$, $i = 1, .., N'$. The impurity potential felt by a particle is the sum of the individual impurity potentials, and we assume for simplicity that the impurities

are identical:

$$V(\mathbf{x}) = \sum_{i=1}^{N'} V_{imp}(\mathbf{x}-\mathbf{r}_i) = \int d\mathbf{x}'\, V_{imp}(\mathbf{x}')\,\rho_{imp}(\mathbf{x}-\mathbf{x}') \tag{3.1}$$

where

$$\rho_{imp}(\mathbf{x}) \equiv \rho_{imp}(\mathbf{x};\mathbf{r}_1,..,\mathbf{r}_{N'}) = \sum_{i=1}^{N'} \delta(\mathbf{x}-\mathbf{r}_i) \tag{3.2}$$

is the impurity density distribution, which has the Fourier transform

$$\rho_{imp}(\mathbf{k}) \equiv \rho_{imp}(\mathbf{k};\mathbf{r}_1,..,\mathbf{r}_{N'}) = \int d\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\,\rho_{imp}(\mathbf{x}) = \sum_{i=1}^{N'} e^{-i\mathbf{k}\cdot\mathbf{r}_i}\ . \tag{3.3}$$

The Fourier transform of the potential is

$$V(\mathbf{q}) = V_{imp}(\mathbf{q}) \sum_{i=1}^{N'} e^{-i\mathbf{q}\cdot\mathbf{r}_i} \tag{3.4}$$

where we have introduced the Fourier transform of the potential of the individual impurity

$$V_{imp}(\mathbf{q}) = \int_V d\mathbf{x}\, e^{-i\mathbf{x}\cdot\mathbf{q}}\, V_{imp}(\mathbf{x}) \tag{3.5}$$

V denoting the volume of the system.

In a large system, where the impurities are evenly distributed, one might expect that the properties of the system would be characterized solely in terms of the macroscopic parameter, the mean impurity concentration $n_i = N'/V$. This is in accordance with the usual statistical description of macroscopic systems in equilibrium, where one assumes that the behavior of the system can be characterized as an average over its ensemble of macroscopically identical subsystems. The sample is said to be self-averaging. The precise conditions under which this macroscopic ensemble point of view provides a sufficient description of the transport properties of a sample is a question which has only been resolved recently. It has been realized that average values are not exhaustive for characterizing a conductor at sufficiently low temperatures. There are important quantum interference effects contributing to transport properties which do not behave in the fashion of thermodynamic fluctuations. For example, in two dimensions the quantum fluctuations in transport properties at zero temperature are independent of the size of the system, and hence do not vanish in the thermodynamic limit. In this case a transport property is not completely characterized by its average value, we need to know higher moments of the distribution as well. We shall discuss these so-called mesoscopic fluctuations in detail in chapter 11, where we will obtain quantitative criteria for the applicability of the macroscopic description.

In each subsystem of a disordered conductor, having N impurities and volume V, we contemplate a mesh of M cells of volumes $\Delta\mathbf{r}_M$, assumed to be so fine that the mean distance between the impurities, $n_i^{-1/d} = (V/N)^{1/d}$, is much larger than

the cell size. The probability for having more than one impurity in each cell is therefore insignificant (of order $1/V^2$).

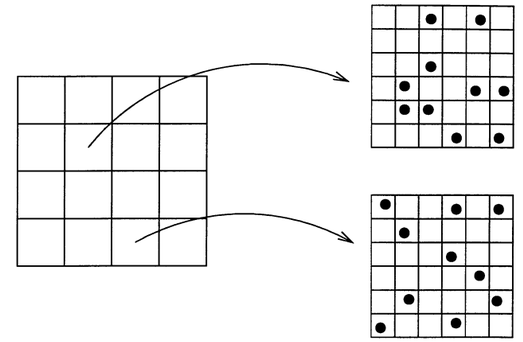


Figure 3.1 Individual impurity configurations of subsamples of a macroscopic sample.

A particular impurity configuration of any subsample is thus equivalent to specifying in which cells the impurities are located: one in $\Delta\mathbf{r}_{i_1}$,.., one in $\Delta\mathbf{r}_{i_N}$. In the limit of small cells the probability, $P(\Delta\mathbf{r}_{i_1},..,\Delta\mathbf{r}_{i_N})$, for this configuration of impurities is specified in terms of a probability density

$$P(\Delta\mathbf{r}_{i_1},..,\Delta\mathbf{r}_{i_N}) = P(\mathbf{r}_{i_1},..,\mathbf{r}_{i_N})\,\Delta\mathbf{r}_{i_1}\cdot\cdot\Delta\mathbf{r}_{i_N} \tag{3.6}$$

where $\mathbf{r}_{i_k}$ is a point in cell k. Assuming that the positions of the impurities are distributed independently, we have for the probability density for having the impurities located at points $\mathbf{r}_1,..,\mathbf{r}_N$

$$P(\mathbf{r}_1,..,\mathbf{r}_N) = P(\mathbf{r}_1)\cdot\cdot\,P(\mathbf{r}_N) = \prod_{i=1}^{N} P(\mathbf{r}_i) \tag{3.7}$$

where $P(\mathbf{r}_i)$ is the probability density for having an impurity in the cell around point $\mathbf{r}_i$.

In the random case where we assume equal probability for an impurity to be located in any cell, we then have by normalization[1]

$$P(\mathbf{r}) = \frac{1}{V} \tag{3.8}$$

the random potential case.

[1]No confusion between the notation for the potential and the volume should arise.

In the self-averaging case, the value of a physical quantity F is represented by the average value over the ensemble of macroscopically identical subsystems. The average value is obtained by taking the average over systems where the location of impurities differ, but they have identical mean impurity concentration. In any given system of this ensemble, each with N impurities, the quantity F will depend on the actual positions of the impurities, $F = F(\mathbf{r}_1, \mathbf{r}_2, ..\mathbf{r}_N)$, and for the impurity average of F we have

$$\begin{aligned} <F> &\equiv \int \prod_{i=1}^{N} d\mathbf{r}_i \, F(\mathbf{r}_1, \mathbf{r}_2, .., \mathbf{r}_N) \, P(\mathbf{r}_1, \mathbf{r}_2, .., \mathbf{r}_N) \\ &= \int \prod_{i=1}^{N} \frac{d\mathbf{r}_i}{V} \, F(\mathbf{r}_1, \mathbf{r}_2, .., \mathbf{r}_N) \end{aligned} \tag{3.9}$$

where the last equality is valid for the random case.

3.2 Propagation in a Random Potential

The all important ingredient in the further analysis is the impurity-averaged propagator. It is the basic building block we shall need when we discuss the motion of a particle in a random potential. Using the results of the previous chapter we study the impurity-averaged propagator order by order in perturbation theory.[2] In a given sample or subsystem the propagator (here displayed in the momentum representation)

$$G^R(\mathbf{p}, t; \mathbf{p}', t') \equiv G^R(\mathbf{p}, t; \mathbf{p}', t'; \mathbf{r}_1, \mathbf{r}_2, ..\mathbf{r}_N) \tag{3.10}$$

depends parametrically on the impurity positions.

The impurity-averaged propagator can be obtained by averaging each term in the perturbative expansion of the propagator in terms of the impurity potential. With each impurity potential we have in the momentum representation, according to eq.(3.4), associated the factor $V(\mathbf{p}-\mathbf{p}')$ depending on the incoming and outgoing momenta to the vertex (in accordance with eq.(2.132)).[3] Expressing the impurity potential in terms of the individual impurity potentials

$$V(\mathbf{p}-\mathbf{p}') \equiv V(\mathbf{p}, \mathbf{p}'; \mathbf{r}_1, \mathbf{r}_2, ..\mathbf{r}_N) = V_{imp}(\mathbf{p}-\mathbf{p}') \sum_{i=1}^{N} e^{-\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}_i} \tag{3.11}$$

we have associated with each potential term the impurity phase factor

$$\rho_{imp}(\mathbf{p}-\mathbf{p}'; \mathbf{r}_1, .., \mathbf{r}_N) = \sum_{i=1}^{N} \exp\left\{-\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}_i\right\} . \tag{3.12}$$

[2] We follow the original presentations of references [19] and [20].

[3] We shall, when convenient, instead of the wave vector representation use the momentum representation of the previous chapter, $V(\mathbf{k}-\mathbf{k}') \to V(\mathbf{p}-\mathbf{p}')$, $\mathbf{p} \equiv \hbar\mathbf{k}$.

Averaging the first-order term, eq.(2.121), we get in the momentum representation[4]

$$<G_1^R(\mathbf{p}, \mathbf{p}', E)> = G_0^R(\mathbf{p}, E) \, V_{imp}(\mathbf{p}-\mathbf{p}') \, G_0^R(\mathbf{p}', E) \frac{1}{V} <\sum_{i=1}^{N} e^{-\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}_i}> \tag{3.13}$$

and we have to impurity average the Fourier transform of the impurity density. When $\mathbf{p} \neq \mathbf{p}'$, the oscillating exponents average to zero, and for $\mathbf{p} = \mathbf{p}'$ we get the number of impurities; i.e.,

$$<\sum_{i=1}^{N} e^{-\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}_i}> = N \, \delta_{\mathbf{p},\mathbf{p}'} \tag{3.14}$$

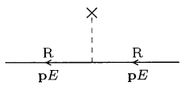
because there are N terms in the sum giving identical contributions. The overall factor in eq.(3.13) is therefore the average impurity density. As expected we have recovered translation invariance for the first-order impurity-averaged propagator

$$< G_1^R(\mathbf{p}, \mathbf{p}', E) > = n_i \, V_{imp}(\mathbf{p}=\mathbf{0}) \, [G_0^R(\mathbf{p}, E)]^2 \, \delta_{\mathbf{p},\mathbf{p}'} . \tag{3.15}$$

The factor in front of the Kronecker function

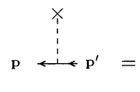
$$G_1^R(\mathbf{p}, E) = n_i \, V_{imp}(\mathbf{p}=\mathbf{0}) [G_0^R(\mathbf{p}, E)]^2 \tag{3.16}$$

we can depict diagrammatically

$G_1^R(\mathbf{p}, E) =$ 

(3.17)

as we introduce



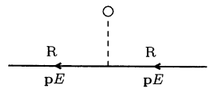
$= n_i \, V_{imp}(\mathbf{p}-\mathbf{p}')$ (3.18)

where the cross designates the impurity concentration, n_i, and the dashed line the Fourier transform of the impurity potential, $V_{imp}(\mathbf{p}-\mathbf{p}')$, where the argument is the outgoing minus the incoming momentum.

The first-order term is proportional to $V_{imp}(\mathbf{p}=\mathbf{0})$, the spatial average of the impurity potential, whose value is arbitrary, and just acts as a constant, $n_i V_{imp}(\mathbf{p}=\mathbf{0})$, added to the Hamiltonian. This term therefore has no observable consequences, and we can assume it to be zero, or redefine the reference for measuring energy.

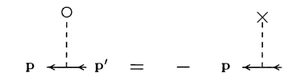
[4] We shall for simplicity assume the free electron model for the conduction electrons. Formally, however, all formulas are identical for Bloch electrons; we just have a suppressed band index. Eventually, one must usually in order to obtain analytical results resort to a simple Fermi surface, and neglect interband scattering.

The latter case is a simple example of renormalizing the term away; i.e., we add the constant $-n_i\, V_{imp}(\mathbf{0})$ to the Hamiltonian, and this term will generate the first-order term



$$= \quad G_0^R(\mathbf{p},E)(-n_i\, V_{imp}(\mathbf{0}))G_0^R(\mathbf{p},E) \tag{3.19}$$

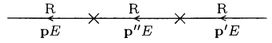
canceling the previous term. The extra diagrams generated by the added term to the Hamiltonian will exactly cancel all the dangling impurity-scattering diagrams as



(3.20)

For the second-order term, eq.(2.127),

$$G_2^R(\mathbf{p},\mathbf{p}',E) \;=$$



(3.21)

we get, upon impurity averaging,

$$< G_2^R(\mathbf{p},\mathbf{p}',E) > \;=\; G_0^R(\mathbf{p},E)\sum_{\mathbf{p}''} V_{imp}(\mathbf{p}-\mathbf{p}'')\, G_0^R(\mathbf{p}'',E)\, V_{imp}(\mathbf{p}''-\mathbf{p}')$$

$$G_0^R(\mathbf{p}',E)\,\frac{1}{V^2}\,\langle \sum_{i,j=1}^{N} e^{-\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}'')\cdot\mathbf{r}_i-\frac{i}{\hbar}(\mathbf{p}''-\mathbf{p}')\cdot\mathbf{r}_j}\rangle \tag{3.22}$$

and we have to average the sum over the impurity positions. The terms in the sum corresponding to scattering off different impurities, $i \neq j$, gives the factor

$$\frac{1}{V^2}\,\langle \sum_{i\neq j}^{N} e^{-\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}'')\cdot\mathbf{r}_i-\frac{i}{\hbar}(\mathbf{p}''-\mathbf{p}')\cdot\mathbf{r}_j}\rangle \;=\; \frac{N(N-1)}{V^2}\,\delta_{\mathbf{p},\mathbf{p}''}\,\delta_{\mathbf{p}'',\mathbf{p}'} \tag{3.23}$$

as there are $N(N-1)$ terms in the sum giving identical contributions. We shall be interested in the thermodynamic limit where we let V and N approach infinity in such a way that the ratio is kept fixed, $n_i \equiv N/V$, and we get

$$\frac{1}{V^2}\,\langle \sum_{i\neq j}^{N} e^{-\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}'')\cdot\mathbf{r}_i-\frac{i}{\hbar}(\mathbf{p}''-\mathbf{p}')\cdot\mathbf{r}_j}\rangle \;=\; n_i^2\,\delta_{\mathbf{p},\mathbf{p}''}\,\delta_{\mathbf{p}'',\mathbf{p}'} \;=\; n_i^2\,\delta_{\mathbf{p},\mathbf{p}''}\,\delta_{\mathbf{p},\mathbf{p}'} \tag{3.24}$$

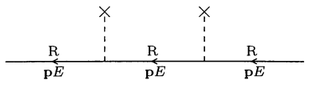
and thereby the contribution from scattering off different impurities

$$< G_2^R(\mathbf{p},\mathbf{p}',E) >^{i\neq j} \;=\; \delta_{\mathbf{p},\mathbf{p}'}\, n_i^2\,[V_{imp}(\mathbf{p}=\mathbf{0})]^2\,[G_0^R(\mathbf{p},E)]^3$$

$$\equiv \; G_2^R(\mathbf{p},E)^{i\neq j}\,\delta_{\mathbf{p},\mathbf{p}'}\,. \tag{3.25}$$

The prefactor in this term has the diagrammatic representation

$$G_2^R(\mathbf{p},E)^{i\neq j} \;=$$



$$= \; [G_0^R(\mathbf{p},E)]^3\, n_i^2\,[V_{imp}(\mathbf{p}=\mathbf{0})]^2 \tag{3.26}$$

with two dangling impurity lines. The term can be kept or renormalized away by the above mentioned prescription.

The term corresponding to scattering off the same impurity $i = j$ gives the factor

$$\langle \sum_{i=1}^{N} e^{-\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}')\cdot\mathbf{r}_i}\rangle \;=\; N\,\delta_{\mathbf{p},\mathbf{p}'} \tag{3.27}$$

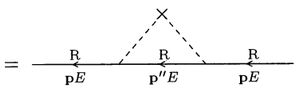
as there are N terms in the sum giving identical contributions, and we get the contribution to the impurity-averaged propagator

$$< G_2^R(\mathbf{p},\mathbf{p}',E) >^{i=j} \;=\; n_i\,\delta_{\mathbf{p},\mathbf{p}'}\,[G_0^R(\mathbf{p},E)]^2\,\frac{1}{V}\sum_{\mathbf{p}''}|V_{imp}(\mathbf{p}-\mathbf{p}'')|^2\,G_0^R(\mathbf{p}'',E)$$

$$\equiv \; G_2^R(\mathbf{p},E)^{i=j}\,\delta_{\mathbf{p},\mathbf{p}'}\,. \tag{3.28}$$

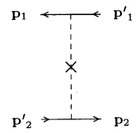
The prefactor in this term has the diagrammatic representation

$$G_2^R(\mathbf{p},E)^{i=j} \;=$$



(3.29)

where we have introduced the impurity correlator in the momentum representation



$$\equiv \; n_i\,\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_1'+\mathbf{p}_2'}\; V_{imp}(\mathbf{p}_1-\mathbf{p}_1')\; V_{imp}(\mathbf{p}_2-\mathbf{p}_2')$$

$$= \; n_i\,\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_1'+\mathbf{p}_2'}\;|V_{imp}(\mathbf{p}_1-\mathbf{p}_1')|^2 \quad . \tag{3.30}$$

The Kronecker function reflects that translation invariance is recovered upon impurity averaging, and expresses that the sum of the incoming momenta in the impurity correlator equals the sum of the outgoing momenta. In the above second-order case the momentum conservation is trivially expressed as $1 = \delta_{\mathbf{p},\mathbf{p}} = \delta_{\mathbf{p}+\mathbf{p}'',\mathbf{p}+\mathbf{p}''}$. Summation (and an inverse volume factor) over the internal momentum variable is implied as a Feynman rule.

To second order in the impurity potential the impurity-averaged propagator has recovered translation invariance and we have in the continuum limit

$$G_2^R(\mathbf{p},E) = [G_0^R(\mathbf{p},E)]^2\, n_i \int \frac{d\mathbf{p}'}{(2\pi\hbar)^3}\, |V_{imp}(\mathbf{p}-\mathbf{p}')|^2 G_0^R(\mathbf{p}',E)\ . \tag{3.31}$$

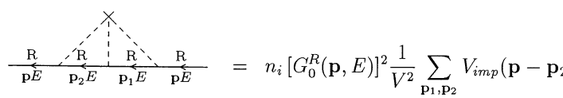
For the third-order diagram

$$G_3^R(\mathbf{p},\mathbf{p}',E) \;=\; \underset{\mathbf{p}E \quad \mathbf{p}_2E \quad \mathbf{p}_1E \quad \mathbf{p}'E}{\overset{R \quad R \quad R \quad R}{\bullet\!\leftarrow\!\times\!\leftarrow\!\times\!\leftarrow\!\times\!\leftarrow\!\bullet}} \tag{3.32}$$

we have the sum of position-dependent impurity phase factors

$$< \sum_{i,j,k=1}^{N} e^{-\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}_2)\cdot\mathbf{r}_i - \frac{i}{\hbar}(\mathbf{p}_2-\mathbf{p}_1)\cdot\mathbf{r}_j - \frac{i}{\hbar}(\mathbf{p}_1-\mathbf{p}')\cdot\mathbf{r}_k} > \tag{3.33}$$

which for triple scattering off the same impurity, $i = j = k$, gives the prefactor $N\delta_{\mathbf{p},\mathbf{p}'}$ and thereby the diagram



$$= n_i\,[G_0^R(\mathbf{p},E)]^2 \frac{1}{V^2} \sum_{\mathbf{p}_1,\mathbf{p}_2} V_{imp}(\mathbf{p}-\mathbf{p}_2) G_0^R(\mathbf{p}_2,E)$$

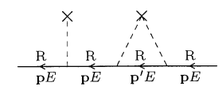
$$V_{imp}(\mathbf{p}_2-\mathbf{p}_1) G_0^R(\mathbf{p}_1,E) V_{imp}(\mathbf{p}_1-\mathbf{p}) \tag{3.34}$$

where the three-leg represents the three impurity potential factors, and the cross, as before, the impurity concentration. Terms with three or more scatterings off the same impurity, we shall refer to as multiple scattering.

The terms with double scattering off the same impurity, the $i \neq j = k$, $i = j \neq k$ terms, contains a dangling impurity line, as the average gives

$$\frac{N(N-1)}{V^2}\, \delta_{\mathbf{p},\mathbf{p}_2}\, \delta_{\mathbf{p}_2,\mathbf{p}'} \;\simeq\; \frac{N^2}{V^2}\, \delta_{\mathbf{p},\mathbf{p}_2}\, \delta_{\mathbf{p},\mathbf{p}'} \tag{3.35}$$

where the last expression is an identity in the large volume limit. For example we have the contribution



$$= n_i V_{imp}(\mathbf{p}=\mathbf{0})[G_0^R(\mathbf{p},E)]^3$$

$$n_i \frac{1}{V} \sum_{\mathbf{p}'} |V_{imp}(\mathbf{p}-\mathbf{p}')|^2 G_0^R(\mathbf{p}',E) \tag{3.36}$$

which can be kept or renormalized to zero.

The fourth-order diagram

$$\underset{\mathbf{p}E \quad \mathbf{p}_3E \quad \mathbf{p}_2E \quad \mathbf{p}_1E \quad \mathbf{p}'E}{\overset{R \quad R \quad R \quad R \quad R}{\leftarrow\!\times\!\leftarrow\!\times\!\leftarrow\!\times\!\leftarrow\!\times\!\leftarrow}} \tag{3.37}$$

has the impurity phase factor

$$< \sum_{i,j,k,l=1}^{N} e^{-\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}_3)\cdot\mathbf{r}_i - \frac{i}{\hbar}(\mathbf{p}_3-\mathbf{p}_2)\cdot\mathbf{r}_j - \frac{i}{\hbar}(\mathbf{p}_2-\mathbf{p}_1)\cdot\mathbf{r}_k - \frac{i}{\hbar}(\mathbf{p}_1-\mathbf{p}')\cdot\mathbf{r}_l} > \ . \tag{3.38}$$

The term corresponding to quadruple scattering off the same impurity, $i = j = k = l$, contains the factor

$$< \sum_{i=j=k=l}^{N} e^{-\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}_3)\cdot\mathbf{r}_i - \frac{i}{\hbar}(\mathbf{p}_3-\mathbf{p}_2)\cdot\mathbf{r}_j - \frac{i}{\hbar}(\mathbf{p}_2-\mathbf{p}_1)\cdot\mathbf{r}_k - \frac{i}{\hbar}(\mathbf{p}_1-\mathbf{p}')\cdot\mathbf{r}_l} > = N\,\delta_{\mathbf{p},\mathbf{p}'} \tag{3.39}$$

giving the multiple scattering term corresponding to the diagram

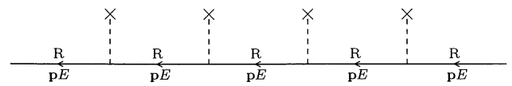
$$G_4^R(\mathbf{p},E)^{i=j=k=l} \;=\; \underset{\mathbf{p}E \quad \mathbf{p}_3E \quad \mathbf{p}_2E \quad \mathbf{p}_1E \quad \mathbf{p}E}{\overset{R \quad R \quad R \quad R \quad R}{\leftarrow\!\leftarrow\!\leftarrow\!\leftarrow\!\leftarrow}}$$

$$= n_i \int \frac{d\mathbf{p}_3}{(2\pi\hbar)^3} \int \frac{d\mathbf{p}_2}{(2\pi\hbar)^3} \int \frac{d\mathbf{p}_1}{(2\pi\hbar)^3}\, V_{imp}(\mathbf{p}-\mathbf{p}_3)$$

$$V_{imp}(\mathbf{p}_3-\mathbf{p}_2) V_{imp}(\mathbf{p}_2-\mathbf{p}_1) V_{imp}(\mathbf{p}_1-\mathbf{p})$$

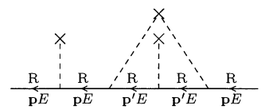
$$G_0^R(\mathbf{p}_3,E) G_0^R(\mathbf{p}_2,E) G_0^R(\mathbf{p}_1,E) [G_0^R(\mathbf{p},E)]^2\ . \tag{3.40}$$

We also get fourth-order terms corresponding to diagrams with dangling impurity lines. For example the $N(N-1)(N-2)(N-3)$ terms where the scattering is off different impurities $i \neq j \neq k \neq l$,



(3.41)

or terms with two dangling lines, for example the term where $i \neq j = l \neq k$,



. (3.42)

All such dangling impurity line diagrams we can keep or renormalize to zero.

Interesting terms arise when we have double scattering off two different impurities – for example, the case where we consider the term $i = j \neq k = l$ in the sum. The impurity phase factor is then

$$< e^{-\frac{i}{\hbar}(\mathbf{p}-\mathbf{p}_2)\cdot\mathbf{r}_i} e^{-\frac{i}{\hbar}(\mathbf{p}_2-\mathbf{p}')\cdot\mathbf{r}_k} > \;=\; \delta_{\mathbf{p},\mathbf{p}_2}\,\delta_{\mathbf{p}_2,\mathbf{p}'} \;=\; \delta_{\mathbf{p},\mathbf{p}_2}\,\delta_{\mathbf{p},\mathbf{p}'}$$
$$= \delta_{\mathbf{p}_3+\mathbf{p}_2,\mathbf{p}_3+\mathbf{p}}\;\delta_{\mathbf{p}_2+\mathbf{p}_1,\mathbf{p}'+\mathbf{p}_1}\,. \tag{3.43}$$

Double scattering off two impurities can occur in three different ways, the considered one $i = j \neq k = l$, and the other two possibilities $i = k \neq j = l$ and $i = l \neq j = k$, and the three contributions correspond to the diagrams

(3.44)

The impurity correlators lead in each term to different momentum-conservation constraints.

Continuing impurity averaging the higher-order terms, the impurity-averaged propagator is expressed as a perturbative expansion in the impurity concentration

or the impurity potential, and the bare propagator, G_0^R, the propagator in the absence of the impurity potential.

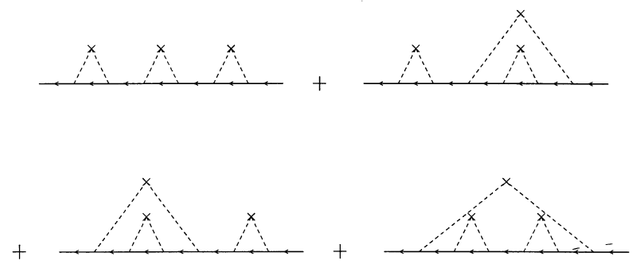
The approximation where multiple scatterings can be neglected we shall call the Born approximation, and we establish the quantitative criterion for its validity in section 3.5. As we discuss in section 3.7, the multiple scattering terms can be taken into account without qualitative changes for the cases we shall have in mind.[5] In the Born approximation we therefore have an easy prescription for obtaining all the diagrams for the n'th order impurity averaged propagator: Tie all the impurity crosses pairwise together in all possible ways! We note, that in the no-dangling Born approximation the perturbation series is an expansion in the parameter $n_i|V_{imp}|^2$.

To each order in perturbation theory we have translation invariance of the impurity-averaged propagator, and we have analytically for the impurity-averaged propagator

$$< G^R(\mathbf{p},\mathbf{p}',E) > \;=\; G^R(\mathbf{p},E)\,\delta_{\mathbf{p}\mathbf{p}'}\,. \tag{3.45}$$

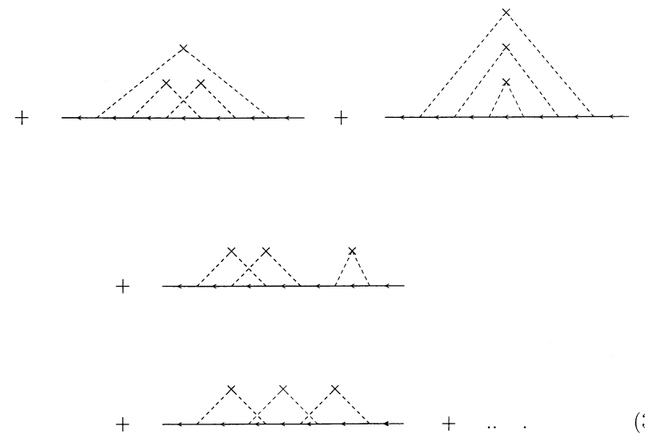
The translation invariance of the impurity-averaged propagator is obvious from a physical point of view because the averaging procedure distinguishes no point. We have verified this property to each order in perturbation theory, as the momentum conservation at the impurity correlator, eq.(3.30), assures that the total momentum flow through a diagram is conserved.[6]

The diagrams of third order in the impurity concentration are in the Born approximation (leaving out all dangling impurity line diagrams) the following



[5] When the potential can give rise to resonances, multiple scattering can give rise to effects not included in the Born approximation.

[6] Such an identification of a property valid in each order of perturbation theory can forcibly be turned around to allow statements valid beyond perturbation theory.



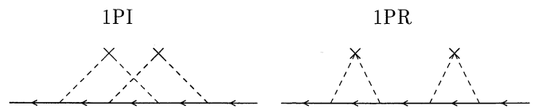
(3.46)

Exercise 3.1 *Draw the rest of the* 15 *third-order diagrams.*

3.3 The Self-energy

We have previously derived diagrammatic formulas from formal expressions. Now we shall argue directly in the diagrammatic language in order to generate new diagrammatic expressions from previous ones, and thereby diagrammatically derive new equations.

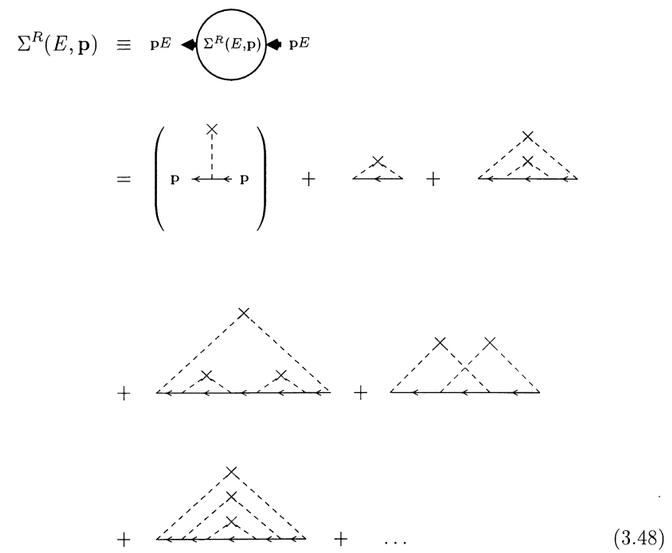
In order to get a grasp of the totality of diagrams for the impurity-averaged propagator we shall use their topology for classification. We introduce the one-particle irreducible (1PI) propagator, corresponding to all the diagrams which can not be cut in two by cutting an internal particle line. In the example



(3.47)

the first diagram is one-particle irreducible,, 1PI, whereas the second is one-particle reducible, 1PR.

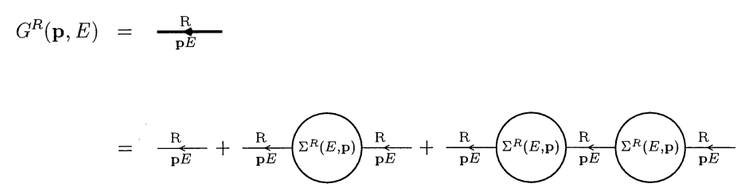
Amputating the external legs of the one-particle irreducible diagrams for the impurity-averaged propagator, we get an object we call the self-energy:

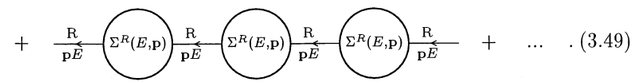


(3.48)

consisting, by construction, of all amputated diagrams which can not be cut in two by cutting one bare propagator line.

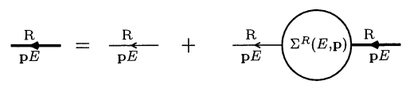
We can now go on and uniquely classify all diagrams of the impurity-averaged propagator according to whether they can be cut in two by cutting an internal particle line at only one place, or at two, three, etc. places. By construction we uniquely exhaust all the possible diagrams for the propagator





(3.49)

By iteration, this equation is seen to be equivalent to the equation[7]



(3.50)

In terms of the self-energy we therefore have for the impurity-averaged propagator the equation

$$G^R(\mathbf{p},E) \;=\; G_0^R(\mathbf{p},E) \;+\; G_0^R(\mathbf{p},E)\,\Sigma^R(E,\mathbf{p})\,G^R(\mathbf{p},E) \tag{3.51}$$

which we can solve to get

$$G^R(\mathbf{p},E) \;=\; \frac{1}{G_0^{-1}(\mathbf{p},E)-\Sigma^R(E,\mathbf{p})} \;=\; \frac{1}{E-\epsilon_{\mathbf{p}}-\Sigma^R(E,\mathbf{p})}\,. \tag{3.52}$$

The self-energy determines the analytic structure of the propagator, the location of the poles of the analytically continued propagator, and thereby the lifetime of (in the present case) momentum states. The effect of the random potential is clearly to give momentum states a finite lifetime (see also exercise 3.3 on page 147).

Fourier transforming eq.(3.52) we get in our isotropic model (in three spatial dimensions for the prefactor to be correct)

$$G_E^R(\mathbf{x}-\mathbf{x}') \;\equiv\; <G_E^R(\mathbf{x},\mathbf{x}')> \;=\; \frac{-m}{2\pi\hbar^2}\,\frac{e^{\frac{i}{\hbar}|\mathbf{x}-\mathbf{x}'|\sqrt{2m(E-\Sigma^R(E,p_E\hat{\mathbf{p}}))}}}{|\mathbf{x}-\mathbf{x}'|} \tag{3.53}$$

where p_E is the solution of the equation

$$p_E = \sqrt{2m(E-\Sigma^R(E,p_E\hat{\mathbf{p}}))}\quad . \tag{3.54}$$

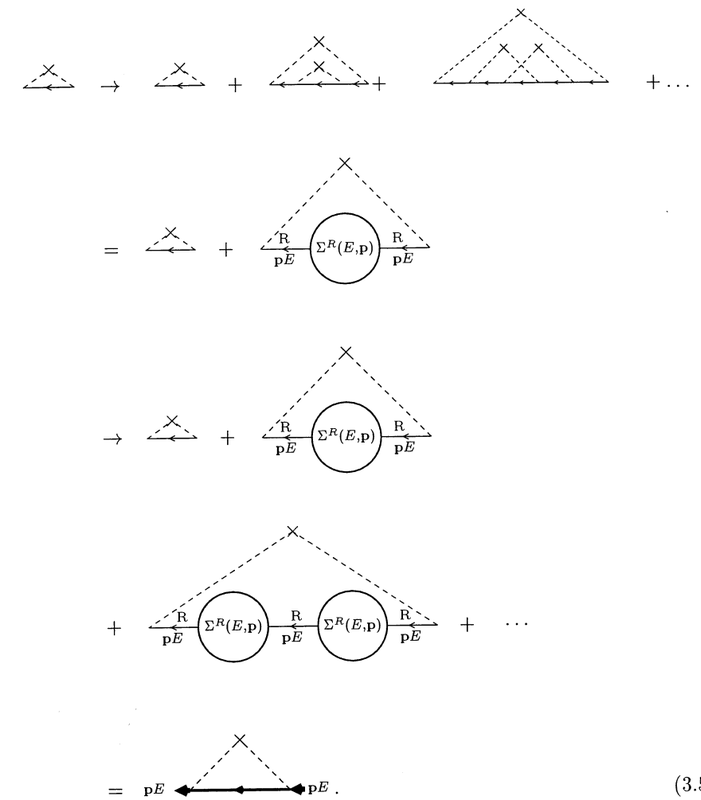
3.4 Skeleton Diagrams

So far we only have a perturbative description of the self-energy; i.e., we have a representation of the self-energy as a functional of the free propagator $\Sigma_{\mathbf{p},E}[G_0^R]$. In a realistic description of a physical system, we always need to invoke the specifics of the problem in order to implement a controlled approximation. To this end

[7] In the last term we can interchange the free and full propagator, because iterating from the left generates the same series as iterating from the right.

we must study the actual correlations in the system, and it is necessary to have the self-energy expressed in terms of the impurity-averaged propagator. Coherent quantum processes correspond to an infinite repetition of bare processes, and the diagrammatic approach is precisely useful for capturing this feature, as irreducible summations are easily described diagrammatically. In order to achieve a description of the self-energy in terms of the full propagator, let us consider the perturbative expansion of the self-energy.

For any given self-energy diagram in the perturbative expansion, we also encounter self-energy diagrams with all possible self-energy decorations on internal lines

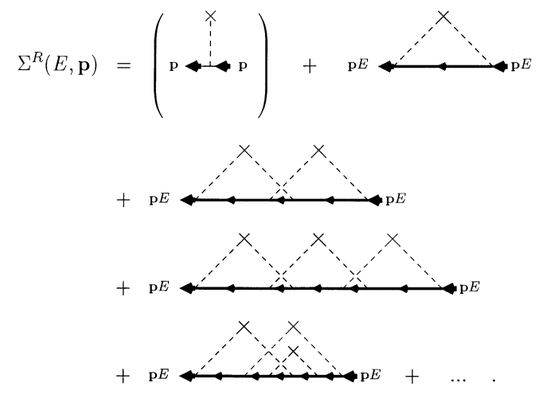


(3.55)

We can uniquely classify all the self-energy diagrams in the perturbative expansion

according to whether the internal particle line can be cut into two, three, or more pieces by cutting particle lines. We can therefore partially sum the self-energy diagrams according to the unique prescription: for a given self-energy diagram, remove all internal self-energy insertions, and substitute for the remaining bare particle propagator lines the full impurity-averaged propagator lines.[8] Through this partial summation of the original perturbative expansion of the self-energy only so-called skeleton diagrams containing the full propagator will then appear. By construction, only skeleton diagrams which can not be cut in two by cutting only two full propagator lines appear, and we have the partially summed diagrammatic expansion for the self-energy[9]

$\Sigma^R(E,\mathbf{p})$ =



(3.56)

What has been achieved by the partial summation, where each diagram corresponds to an infinite sum of terms in perturbation theory, is that the self-energy is expressed as a functional of the exact impurity-averaged propagator

$$\Sigma^R(\mathbf{p},E) = \Sigma^R_{\mathbf{p},E}[G^R] \; . \tag{3.57}$$

We can continue this topological classification, and introduce the higher order vertex functions; however, we defer this until chapter 8.

Exercise 3.2 *Draw the rest of the* 4 *skeleton self-energy diagrams with three impurity correlators.*

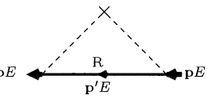
[8] Synonymous names for the full Green's function or propagator are renormalized or dressed propagator.

[9] Since propagator and impurity lines appear topologically on the same footing, we can restate: in the skeleton self-energy expansion only two-line irreducible diagrams occur.

3.5 Impurity-Averaged Propagator

In the next section we show that, for sufficiently high energies and momenta, we can for the self-energy neglect skeleton diagrams where impurity lines cross. We are thereby left with a single self-energy diagram[10]

$\Sigma^R_{NCA}(E,\mathbf{p})$ $\equiv$



(3.58)

in the Born approximation.[11] Analytically we have for the self-energy in the Born approximation

$$\Sigma^R_{NCA}(\mathbf{p},E) = n_i \int \frac{d\mathbf{p}'}{(2\pi\hbar)^3} \, |V_{imp}(\mathbf{p}-\mathbf{p}')|^2 \, G^R(\mathbf{p}',E) \tag{3.59}$$

which is an implicit expression since the propagator is in turn specified in terms of the self-energy.

In order to orient ourselves as to the effect of impurity scattering, we insert the free propagator into eq.(3.59) and obtain

$\tilde{\Sigma}^R_0(\mathbf{p},E)$ $\equiv$

$= -i \, \dfrac{\hbar}{2\,\tau(E,\mathbf{p})}$ $+$ $\Re e\tilde{\Sigma}^R_0(\mathbf{p},E)$ (3.60)

where we have introduced the momentum relaxation time (in the Born approximation)

$$\frac{\hbar}{\tau(E,\mathbf{p})} = 2\pi n_i N_0(E) \int \frac{d\hat{\mathbf{p}}'}{4\pi} \, |V_{imp}(\mathbf{p}-p_E\hat{\mathbf{p}}')|^2 \tag{3.61}$$

where $p_E = \sqrt{2mE}$, and the real part of the self-energy is the principal value integral

$$\Re e\tilde{\Sigma}^R_0(\mathbf{p},E) \; = \; n_i \, P \int_0^\infty d\epsilon_{p'} \; N(\epsilon_{p'}) \, \frac{1}{E-\epsilon_{p'}} \int \frac{d\hat{\mathbf{p}}'}{4\pi} \, |V_{imp}(\mathbf{p}-\sqrt{2m\epsilon_{p'}}\,\hat{\mathbf{p}}')|^2 \tag{3.62}$$

where the P signifies that the principal value of the integral is to be taken. In this first iteration we thus have for the impurity-averaged propagator

$$\tilde{G}^R_0(E,\mathbf{p}) \; = \; \frac{1}{E-\epsilon_p - \Re e\tilde{\Sigma}^R_0(\mathbf{p},E) + i\hbar/2\tau(E,\mathbf{p})} \; . \tag{3.63}$$

In the following we shall only be interested in the region of large energies $E \simeq E_F \gg \hbar/\tau$ and large momenta $p \simeq p_F \equiv \sqrt{2mE_F} \gg \hbar/l$, where $\tau \equiv \tau(E_F,\mathbf{p}_F)$,

[10] Here the index NCA simply stands for noncrossing approximation, but as mentioned, we establish its validity in the next section.

[11] In section 3.6 we show that inclusion of multiple scattering is handled with equal care.

and we have introduced the impurity mean free path $l \equiv v_F\tau$, and the velocity $v_F \equiv p_F/m$.[12] We shall further assume that the impurity potential has a range, a, much shorter than the mean free path, $a \ll l$, say[13]

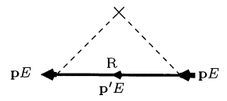
$$V_{imp}(\mathbf{r}) \propto \frac{1}{r}e^{-r/a} \tag{3.64}$$

or for the Fourier transform

$$V_{imp}(\mathbf{k}) \propto \frac{4\pi}{\mathbf{k}^2 + a^{-2}} \ . \tag{3.65}$$

In the region of energies $|E - E_F| \ll E_F$, and momenta $|p - p_F| \ll p_F$, the other relevant quantities for the calculation of the self-energy, density of states and impurity potential, are essentially constant, since they vary only on the large scales E_F and p_F, say $a \simeq \hbar/p_F$. In the region of interest, the real part of the self-energy is thus essentially constant, only giving rise to an irrelevant shift in the reference for measuring the energy.[14] The first iterated propagator expression, eq.(3.63), is therefore as a function of momentum sharply peaked at the value p_F with the small width $\hbar/l$. The result for the imaginary part of the self-energy, eq.(3.61), is therefore unchanged, to order $\hbar/E_F\tau$, by the substitution of the improved propagator, $\tilde{G}_0^R$, instead of the free propagator. We therefore have, to order $\hbar/E_F\tau$, for the self-energy in the Born approximation, $|E - E_F| \ll E_F$, $|p - p_F| \ll p_F$,

$\Sigma^R(E,\mathbf{p})$ $\equiv$



$$= -\frac{i\hbar}{2\tau} \ . \tag{3.66}$$

where

$$\frac{\hbar}{\tau} = 2\pi n_i N_0(E_F) \int \frac{d\hat{\mathbf{p}}'}{4\pi} \left| V_{imp}\left(\sqrt{2mE_F}\,(\hat{\mathbf{p}} - \hat{\mathbf{p}}')\right)\right|^2 \ . \tag{3.67}$$

For the impurity-averaged propagator we therefore obtain, $|E - E_F| \ll E_F$, $|p - p_F| \ll p_F$,

$$< G^R(E,\mathbf{p},\mathbf{p}') > \ = \ G^R(E,\mathbf{p})\,\delta_{\mathbf{p},\mathbf{p}'} \tag{3.68}$$

where

$$G^R(E,\mathbf{p}) = \frac{1}{E - \epsilon_{\mathbf{p}} + i\hbar/2\tau} \ . \tag{3.69}$$

[12]The present single-particle problem has of course no built-in energy scale. However, when we eventually shall discuss the transport properties of degenerate fermions, such as electrons in a metal, say, a large energy scale will be provided, viz. the Fermi energy.

[13]We show in section that this is indeed the relevant case.

[14]For a δ-correlated impurity potential $< V(\mathbf{x})V(\mathbf{x}') > = u^2\,\delta(\mathbf{x} - \mathbf{x}')$ (see also eq.(3.100)), the real part of the self-energy $\Re e\Sigma^R(E,\mathbf{p})$, though divergent, can be absorbed into an irrelevant renormalization of the energy.

We note that this form of the propagator can be used for all values of E and $\mathbf{p}$ without violating the sum rules eq.(2.157) and eq.(2.158).

In the region of interest, $E \simeq E_F$, we have according to eq.(3.53) that the impurity-averaged propagator decays exponentially as a function of its spatial variable with the scale of the mean free path, and is in three spatial dimensions given by the expression

$$G_E^{R(A)}(\mathbf{x} - \mathbf{x}') \equiv < G_E^{R(A)}(\mathbf{x},\mathbf{x}') > = -\frac{m}{2\pi\hbar^2}\frac{\exp\{|\mathbf{x} - \mathbf{x}'|(\stackrel{+}{(-)}\, ik_E - 1/2l_E)\}}{|\mathbf{x} - \mathbf{x}'|}$$

$$= G_0^{R(A)}(\mathbf{x} - \mathbf{x}', E)\, e^{-|\mathbf{x}-\mathbf{x}'|/2l} \tag{3.70}$$

where $k_E = \sqrt{2mE}/\hbar$ is the electronic wave vector, and $l_E = \hbar k_E\tau_E/m$ the impurity mean free path. The imaginary part of the self-energy leads to exponential damping of the propagator as a function of spatial separation. In the present case the damping of the propagator is due to the directional scattering of a plane wave (the spatial representation of the momentum eigenstate) due to the impurities, (say by passing a slab of material with impurities the amplitude of a plane wave is damped).

According to eq.(3.70) and eq.(2.157) the density of states is unchanged by the presence of weak disorder[15]:

$$N(E_F) = N_0(E_F)\,(1 + \mathcal{O}\,(\hbar/E_F\tau)) \ . \tag{3.71}$$

Exercise 3.3 *Show that for $p \simeq p_F$ we have (to order $\hbar/E_F\tau$)*

$$G^R(\mathbf{p},t) = \int_{-\infty}^{\infty}\frac{dE}{2\pi}\, e^{-\frac{i}{\hbar}Et}\,\frac{1}{E - \epsilon_{\mathbf{p}} + \frac{i\hbar}{2\tau(E,p)}} = G_0^R(\mathbf{p},t)\, e^{-t/2\tau} \tag{3.72}$$

i.e., τ is the momentum relaxation time.

3.6 Diagram Estimation

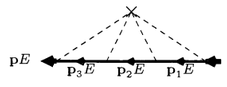
In the skeleton expansion all internal propagators are the exact impurity-averaged propagator instead of the bare propagator as in the naive perturbation expansion in the potential. This is advantageous since it is the full propagator that reflects the physical properties of the system. The properties of the full propagator are determined by its analytic structure, its poles. When we wish to estimate the order of magnitude of the contribution of various diagrams, the quantity of interest to

[15]The specific heat of a degenerate Fermi gas is proportional to the density of states at the Fermi surface. The presence of weak disorder does therefore not change this result, which is not surprising in view of the scattering being elastic.

estimate is therefore the self-energy. This is the quantity determining the pole in the propagator, the singular point where a small change in variables makes a huge difference.

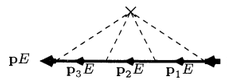
We shall only be interested in estimating the various contributions to the propagator for large energies and momenta, $p \simeq p_F \gg \hbar/l$, $E \simeq E_F \gg \hbar/\tau$. For $E \simeq E_F$, the impurity-averaged propagator, eq.(3.69), is therefore only large near the momentum value p_F where we have for the order of magnitude of the propagator $G \sim \tau/\hbar$, as we have in the Born approximation for the order of magnitude of the self-energy $\Sigma \sim \hbar/\tau$, a small value compared to E_F. For conduction electrons in a metal the large momentum value p_F is the Fermi momentum, and we shall in the following refer to the surface in the space of momentum values at the large momentum value p_F as the Fermi surface. The large contribution to a diagram therefore comes from the internal momentum integration regions where the momenta of the propagators are all on the Fermi surface.

Let us start the diagram estimation by establishing the criterion for the validity of neglecting multiple scattering. Consider for example the fourth-order (in the impurity potential) diagram for the skeleton self-energy



$$= n_i \int \frac{d\mathbf{p}_3}{(2\pi\hbar)^3} \int \frac{d\mathbf{p}_2}{(2\pi\hbar)^3} \int \frac{d\mathbf{p}_1}{(2\pi\hbar)^3} V_{imp}(\mathbf{p}-\mathbf{p}_3) V_{imp}(\mathbf{p}_3-\mathbf{p}_2) V_{imp}(\mathbf{p}_2-\mathbf{p}_1) V_{imp}(\mathbf{p}_1-\mathbf{p}) G^R(\mathbf{p}_3,E) G^R(\mathbf{p}_2,E) G^R(\mathbf{p}_1,E) \quad (3.73)$$

The impurity concentration appears to first order, giving for one of the integrations a self-energy type contribution, and since the Fourier transform of the impurity potential is assumed slowly varying we have the estimate

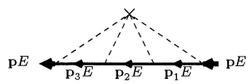


$$\sim \left[V_{imp}(\mathbf{p}=\mathbf{0}) \int \frac{d\mathbf{p}'}{(2\pi\hbar)^3} G^R(\mathbf{p}',E) \right]^2 n_i \int \frac{d\mathbf{p}_1}{(2\pi\hbar)^3} |V_{imp}(\mathbf{p}_1-\mathbf{p})|^2 G^R(\mathbf{p}_1,E) \, . \quad (3.74)$$

For the momentum integral of the impurity-averaged propagator we have

$$\int \frac{d\mathbf{p}}{(2\pi\hbar)^3} G^R(\mathbf{p},E) \sim i N_0(E_F) \sim \frac{m p_F}{\hbar^3} \quad (3.75)$$

and we obtain the estimate

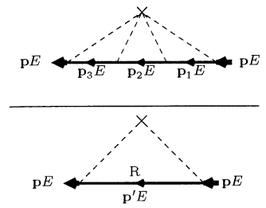


$$\sim \left(\frac{V_{imp}(\mathbf{p}=\mathbf{0}) p_F^2}{\hbar^3 v_F} \right)^2 \frac{\hbar}{\tau} \, . \quad (3.76)$$

In the case where the range of the impurity potential is specified in terms of p_F, $a \sim \hbar/p_F$, the Born criterion is (see for example reference [13])

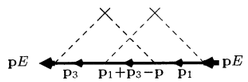
$$p_F^3 \, V_{imp}(\mathbf{p}=\mathbf{0}) \ll \hbar^3 \, E_F \, . \quad (3.77)$$

We therefore have that the multiple scattering term is small relative to the Born term when the Born criterion is satisfied



$$\sim \left(\frac{V_{imp}(\mathbf{p}=\mathbf{0}) p_F^2}{\hbar^3 v_F} \right)^2 \ll \left(\frac{E_F}{v_F} \right)^2 \frac{1}{p_F^2} \sim 1 \, . \quad (3.78)$$

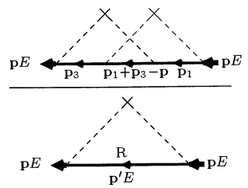
To show that the crossed self-energy diagrams can be neglected when calculating the propagator, we note that the intermediate momentum integrations in a self-energy diagram with crossed impurity lines ($E \simeq E_F, p \simeq p_F$)



$$= n_i^2 \int \frac{d\mathbf{p}_3}{(2\pi\hbar)^3} \int \frac{d\mathbf{p}_1}{(2\pi\hbar)^3} |V_{imp}(\mathbf{p}-\mathbf{p}_3)|^2 |V_{imp}(\mathbf{p}_1-\mathbf{p})|^2 G^R(\mathbf{p}_3,E) G^R(\mathbf{p}_1+\mathbf{p}_3-\mathbf{p},E) G^R(\mathbf{p}_1,E) \quad (3.79)$$

for the region of large contribution, are not free. In order for all the momenta of the propagators to be in the thin shell of extension $\hbar/l$ around the sphere in momentum space with radius p_F, where the large contribution arises, one of the angular integrations is restricted to a cone of angle $\hbar/p_F l$. The crossed diagrams will therefore be relatively smaller by the same factor compared to the diagram where impurity lines do not cross and no angular restriction occurs[16]

[16]The argument being based on an angular integration restriction is thus not valid for the case of one spatial dimension.

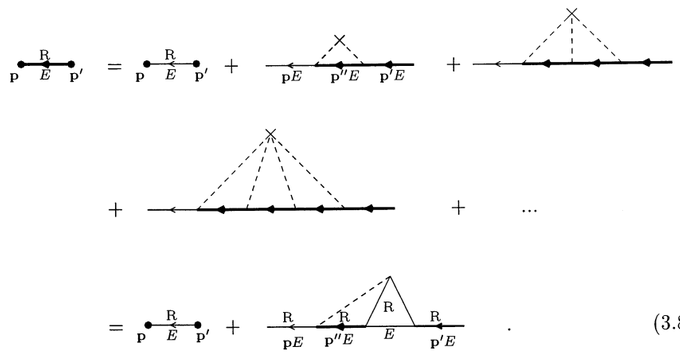


$$\sim \quad \frac{\hbar}{p_F l} \,. \tag{3.80}$$

Using the diagrammatic technique we have thus achieved the goal of identifying a parameter on which we can base a perturbation theory. A nontrivial perturbation expansion as the expansion parameter $\hbar/p_F l$ is not a parameter in the Hamiltonian.

3.7 Multiple Scattering

If we relax the Born restriction, eq.(3.77), we would to order $\hbar/p_F l \ll 1$ have for the impurity-averaged propagator

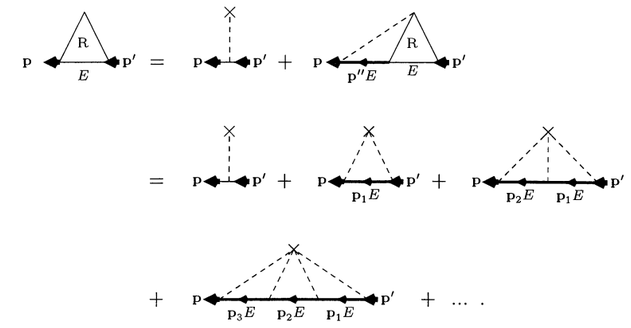


(3.81)

The result beyond the Born approximation is simply obtained by summing all the multiple scatterings, and we will everywhere instead of the Born amplitude $V_{imp}(\mathbf{p}-\mathbf{p}')$ encounter the t-matrix for scattering off a single impurity, where the t-matrix satisfies the equation

$$t^R_{\mathbf{p}\mathbf{p}'}(E) = n_i V_{imp}(\mathbf{p}-\mathbf{p}') + \frac{1}{V}\sum_{\mathbf{p}''} V_{imp}(\mathbf{p}-\mathbf{p}'')\, G^R(\mathbf{p}'',E)\, t^R_{\mathbf{p}''\mathbf{p}'}(E) \tag{3.82}$$

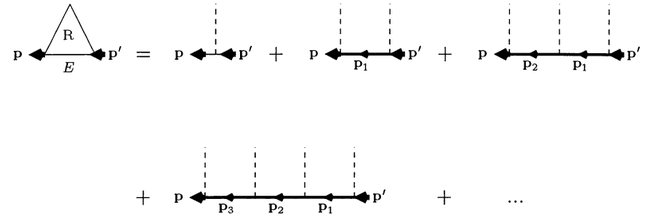
corresponding to the diagrams for the t-matrix



(3.83)

The above t-matrix equation differs from the one for scattering off a single impurity in free space in that the impurity-averaged propagator appears instead of the free propagator, reflecting the presence of the other impurities.

Except for the obvious connection to the previous notation there is no logic to tying the potential lines together since we are concerned with multiple scattering off a single impurity, so we could equally well use the depiction



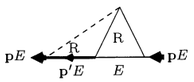
$$= V_{imp}(\mathbf{p}-\mathbf{p}') + \frac{1}{V}\sum_{\mathbf{p}_1} V_{imp}(\mathbf{p}-\mathbf{p}_1)\, G^R(\mathbf{p}_1,E)\, V_{imp}(\mathbf{p}_1-\mathbf{p}')$$

$$+ \frac{1}{V^2}\sum_{\mathbf{p}_1,\mathbf{p}_2} V_{imp}(\mathbf{p}-\mathbf{p}_2)\, G^R(\mathbf{p}_2,E)\, V_{imp}(\mathbf{p}_2-\mathbf{p}_1)$$

$$G^R(\mathbf{p}_1,E)\, V_{imp}(\mathbf{p}_1-\mathbf{p}') + \ldots \tag{3.84}$$

and extract the factor of n_i from the previous definition of the t-matrix. For the self-energy we then have in the noncrossing approximation

$$\Sigma^R(\mathbf{p},E) = \frac{n_i}{V}\sum_{\mathbf{p}'} V_{imp}(\mathbf{p}-\mathbf{p}')\, G^R(\mathbf{p}',E)\, t^R_{\mathbf{p}'\mathbf{p}}(E)$$



$$= \qquad (3.85)$$

and for the imaginary part

$$\Im m\Sigma^R(\mathbf{p},E) \;=\; n_i\,\Im m\,t^R_{\mathbf{pp}}(E)\;. \qquad (3.86)$$

Similarly we encounter when we consider the advanced propagator

$$t^A_{\mathbf{pp}'}(E) = n_i\,V_{imp}(\mathbf{p}-\mathbf{p}') + \frac{1}{V}\sum_{\mathbf{p}''} t^A_{\mathbf{pp}''}(E)\,G^A(\mathbf{p}'',E)\,V_{imp}(\mathbf{p}''-\mathbf{p}') \qquad (3.87)$$

corresponding to the same set of diagrams, but with the impurity-averaged advanced propagator appearing instead of the retarded

$$. \qquad (3.88)$$

We note the hermitian property

$$t^A_{\mathbf{pp}'}(E) = [t^R_{\mathbf{p}'\mathbf{p}}(E)]^* \qquad (3.89)$$

and as always

$$\Sigma^A(\mathbf{p},E) = [\Sigma^R(\mathbf{p},E)]^*\;. \qquad (3.90)$$

Combining the two t-matrix equations we get (extracting the factor n_i from the definition of the t-matrix)

$$t^R_{\mathbf{pp}'}(E) \;=\; V_{imp}(\mathbf{p}-\mathbf{p}') \;+\; \frac{1}{V}\sum_{\mathbf{p}''} t^A_{\mathbf{pp}''}(E)\,G^R(\mathbf{p}'',E)\,t^R_{\mathbf{p}''\mathbf{p}'}(E)$$

$$-\;\frac{1}{V^2}\sum_{\mathbf{p}_1\mathbf{p}_2} t^A_{\mathbf{pp}_1}(E)\,G^A(\mathbf{p}_1,E)\,V_{imp}(\mathbf{p}_1-\mathbf{p}_2)\,G^R(\mathbf{p}_2,E)\,t^R_{\mathbf{p}_2\mathbf{p}'}(E) \qquad (3.91)$$

and similarly

$$t^A_{\mathbf{pp}'}(E) \;=\; V_{imp}(\mathbf{p}-\mathbf{p}') \;+\; \frac{1}{V}\sum_{\mathbf{p}''} t^A_{\mathbf{pp}''}(E)\,G^A(\mathbf{p}'',E)\,t^R_{\mathbf{p}''\mathbf{p}'}(E)$$

$$-\;\frac{1}{V^2}\sum_{\mathbf{p}_1\mathbf{p}_2} t^A_{\mathbf{pp}_1}(E)\,G^A(\mathbf{p}_1,E)\,V_{imp}(\mathbf{p}_1-\mathbf{p}_2)\,G^R(\mathbf{p}_2,E)\,t^R_{\mathbf{p}_2\mathbf{p}'}(E)\;. \qquad (3.92)$$

Subtracting the diagonal terms we get

$$\Im m\,t^R_{\mathbf{pp}}(E) \;=\; -\frac{i}{2}[t^R_{\mathbf{pp}}(E)-t^A_{\mathbf{pp}}(E)] \;=\; \frac{1}{V}\sum_{\mathbf{p}'}|t^R_{\mathbf{pp}'}(E)|^2\,\Im m G^R(\mathbf{p}',E) \qquad (3.93)$$

and thereby for the imaginary part of the self-energy

$$\Im m\Sigma^R(\mathbf{p},E) \;=\; \frac{n_i}{V}\sum_{\mathbf{p}'}|t^R_{\mathbf{pp}'}(E)|^2\,\Im m G^R(\mathbf{p}',E)\;. \qquad (3.94)$$

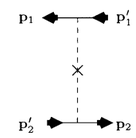
Assuming that the t-matrix is a slowly varying function of momentum on the scale $\hbar/l$ (i.e., no resonances), we get to lowest order in $\hbar/pl$

$$\Im m\Sigma^R(\mathbf{p},E) \;=\; -\frac{\hbar}{2\tau(\mathbf{p},E)} \qquad (3.95)$$

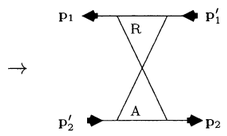
where ($p' = \sqrt{2mE}$)

$$\frac{\hbar}{\tau(\mathbf{p},E)} \;=\; 2\pi n_i N_0(E)\int\frac{d\hat{\mathbf{p}}'}{4\pi}\,|t^R_{\mathbf{pp}'}(E)|^2\;. \qquad (3.96)$$

The only change from the Born approximation, eq.(3.61), being that the exact differential cross section for scattering off an impurity appears instead of the Born expression



$$\equiv\; n_i\,|V_{imp}(\mathbf{p}_1-\mathbf{p}_1')|^2\,\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_1'+\mathbf{p}_2'}$$



$$\rightarrow \qquad \equiv\; n_i\,|t^R_{\mathbf{p}_1-\mathbf{p}_1'}(E)|^2\,\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_1'+\mathbf{p}_2'}\;. \qquad (3.97)$$

Going beyond the Born approximation thus adds nothing qualitatively new if there are no resonances. Furthermore, the topological structure of the diagrams of the disorder problem is also unchanged by including multiple scattering off the same impurity, as the t-matrix correlator simply appears instead of the Gaussian impurity correlator.

3.8 Gaussian Approximation

According to the analysis of impurity averaging the propagator in momentum space, we realize that if we can neglect multiple scattering, and are considering the thermodynamic limit, $(N-1)/V \simeq N/V$, the impurity average corresponds to

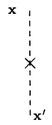
tying potential vertices pairwise together, through the impurity correlator, in all possible ways. In this approximation the impurity average is a Gaussian average in the random variable, the potential. Fourier-transforming, we find that in the position representation the Gaussian average is done according to the specification (for an even number of potential terms)

$$< V(\mathbf{x}_1)V(\mathbf{x}_2) \,..\, V(\mathbf{x}_{2N-1})V(\mathbf{x}_{2N}) >$$

$$= \sum_P < V(\mathbf{x}_{P(1)})V(\mathbf{x}_{P(2)}) > \,..\,.. < V(\mathbf{x}_{P(2N-1)})V(\mathbf{x}_{P(2N)}) > \tag{3.98}$$

where we sum over all permutations P. We have assumed that the mean value of the potential is zero, $< V(\mathbf{x}) > = 0$, so that only fully contracted terms are nonvanishing (diagrams with dangling impurity lines vanish). For an odd number of potential vertices, we always encounter a dangling impurity line, and the contribution is proportional to the mean value of the potential, which we have chosen to be zero.

When impurity averaging a diagram in the position representation, we therefore get the sum of diagrams with all possible pairings of potential vertices by the impurity correlator in real space



$$= \frac{1}{\hbar^2} < V(\mathbf{x})V(\mathbf{x}') > \tag{3.99}$$

which is specified by

$$< V(\mathbf{x})V(\mathbf{x}') > = \frac{1}{V^2}\sum_{\mathbf{p},\mathbf{p}'} e^{\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{x}')}\, V_{imp}(\mathbf{p})\, V_{imp}(\mathbf{p}')\, \langle \sum_{i,j=1}^{N} e^{-\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}_i+\mathbf{p}'\cdot\mathbf{r}_j)} \rangle$$

$$= \frac{N}{V^2}\sum_{\mathbf{p}} e^{\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}|V_{imp}(\mathbf{p})|^2 + \frac{N(N-1)}{V^2}|V_{imp}(\mathbf{p}=\mathbf{0})|^2$$

$$= \frac{N}{V^2}\sum_{\mathbf{p}} e^{\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}|V_{imp}(\mathbf{p})|^2$$

$$= n_i \int d\mathbf{r}\; V_{imp}(\mathbf{x}-\mathbf{r})V_{imp}(\mathbf{x}'-\mathbf{r}) \; . \tag{3.100}$$

where we in the third equality have used that our reference for measuring energies is such that the spatial average of the potential vanishes. Inversely we have

$$< V(\mathbf{p}_1-\mathbf{p}_1')V(\mathbf{p}_2-\mathbf{p}_2') > = \int d\mathbf{x}\int d\mathbf{x}'\; e^{-\frac{i}{\hbar}\mathbf{x}\cdot(\mathbf{p}_1-\mathbf{p}_1')-\frac{i}{\hbar}\mathbf{x}'\cdot(\mathbf{p}_2-\mathbf{p}_2')} < V(\mathbf{x})V(\mathbf{x}') >$$

$$= N\,|V_{imp}(\mathbf{p}_1-\mathbf{p}_1')|^2\,\delta_{\mathbf{p}_1+\mathbf{p}_2,\mathbf{p}_1'+\mathbf{p}_2'} \tag{3.101}$$

the impurity correlator in the momentum representation.

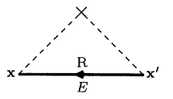
A δ-correlated impurity potential

$$< V(\mathbf{x})\,V(\mathbf{x}') > \;=\; u^2\,\delta(\mathbf{x}-\mathbf{x}') \tag{3.102}$$

corresponds, according to eq.(3.100), to the limit of dense point scatterers.

The self-energy in the Born approximation is in the position representation specified in terms of the impurity correlator

$$\Sigma^R(\mathbf{x},\mathbf{x}';E) = \Sigma^R(\mathbf{x}-\mathbf{x}';E) \equiv$$



$$= < V(\mathbf{x})V(\mathbf{x}') > G^R(\mathbf{x},\mathbf{x}';E) \; . \tag{3.103}$$

In the momentum representation we encountered products of propagators and self-energies. Upon Fourier-transforming, these turn into convolutions in the position variable.

3.9 Motion in a Random Potential

We now have all the ingredients enabling us to perform perturbative calculations for the motion of a particle in a random potential. We introduce the impurity-averaged density matrix in the position representation

$$f(\mathbf{x},\mathbf{x}',t) \equiv <\rho(\mathbf{x},\mathbf{x}',t)> = <<\mathbf{x}|\hat{\rho}(t)|\mathbf{x}'>> \; . \tag{3.104}$$

A diagonal element has the interpretation: Start a particle off at the same point in each impurity sample; then $f(\mathbf{x},\mathbf{x},t)$ gives the probability that we by an arbitrary pick of sample will find a particle at position $\mathbf{x}$ at time t.

In the following we shall obtain a perturbative description of the time evolution of the impurity-averaged density matrix in terms of an integral equation. Let us assume that at time t' the particle is described by the statistical operator $\hat{\rho}(t')$. The impurity-averaged density matrix at time t

$$f(\mathbf{x},\mathbf{x}',t) = \int d\tilde{\mathbf{x}}\int d\tilde{\mathbf{x}}' < J(\mathbf{x},\mathbf{x}',t;\tilde{\mathbf{x}},\tilde{\mathbf{x}}',t') > \rho'(\tilde{\mathbf{x}},\tilde{\mathbf{x}}') \tag{3.105}$$

is then expressed in terms of the density matrix at time t'

$$\rho'(\tilde{\mathbf{x}},\tilde{\mathbf{x}}') \equiv \rho(\tilde{\mathbf{x}},\tilde{\mathbf{x}}',t') = <\tilde{\mathbf{x}}|\hat{\rho}(t')|\tilde{\mathbf{x}}'> \tag{3.106}$$

and the impurity-averaged density-matrix propagator equals

$$< J(\mathbf{x},\mathbf{x}',t;\tilde{\mathbf{x}},\tilde{\mathbf{x}}',t') > = < G_0^R(\mathbf{x},t;\tilde{\mathbf{x}},t')G_0^A(\tilde{\mathbf{x}}',t';\mathbf{x}',t) >$$

$$= \sum_{n=0}^{\infty}(\frac{1}{\hbar})^n \sum_{m=0}^{\infty}(\frac{1}{\hbar})^m \int\prod_{n'=1}^{n} dx_{n'} \int_{-\infty}^{\infty} \prod_{n'=1}^{n} dt_{n'} \int\prod_{m'=1}^{m} d\tilde{x}_{m'} \int_{-\infty}^{\infty} \prod_{m'=1}^{m} d\tilde{t}_{m'}$$

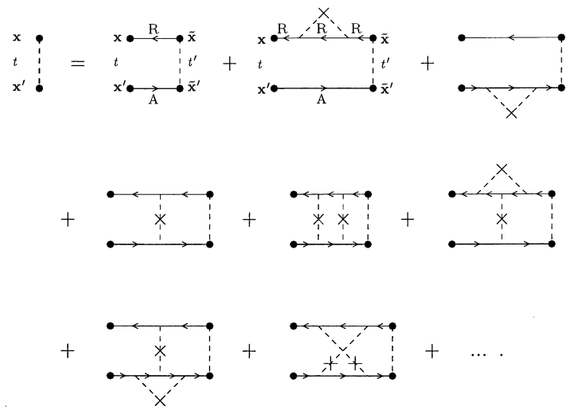
$$G_0^R(\mathbf{x},t;\mathbf{x}_n,t_n)G_0^R(\mathbf{x}_n,t_n;\mathbf{x}_{n-1},t_{n-1})..G_0^R(\mathbf{x}_1,t_1;\tilde{\mathbf{x}},t')$$

$$G_0^A(\tilde{\mathbf{x}}',t';\tilde{\mathbf{x}}_m,\tilde{t}_m)G_0^A(\tilde{\mathbf{x}}_m,\tilde{t}_m;\tilde{\mathbf{x}}_{m-1},\tilde{t}_{m-1})..G_0^A(\tilde{\mathbf{x}}_1,\tilde{t}_1;\mathbf{x}',t)$$

$$< V(\tilde{\mathbf{x}}_m)V(\mathbf{x}_{m-1})..V(\tilde{\mathbf{x}}_1)V(\mathbf{x}_n)V(\mathbf{x}_{n-1})..V(\mathbf{x}_1) > \qquad (3.107)$$

where we have assumed $t > t'$.[17] A virtue of the diagrammatic approach is that we shall not have to deal with unwieldy perturbative expressions as the above!

Before the impurity average has been performed, the perturbative expansion of the density matrix corresponds to the sum of diagrams with any number of potential vertices on the upper and lower lines as depicted in eq.(2.246). In the approximation where we neglect multiple scattering we get the impurity-averaged density matrix by tying all the potential vertices pairwise together, the Gaussian average, and obtain for the impurity-averaged density matrix the diagrammatic expansion



(3.108)

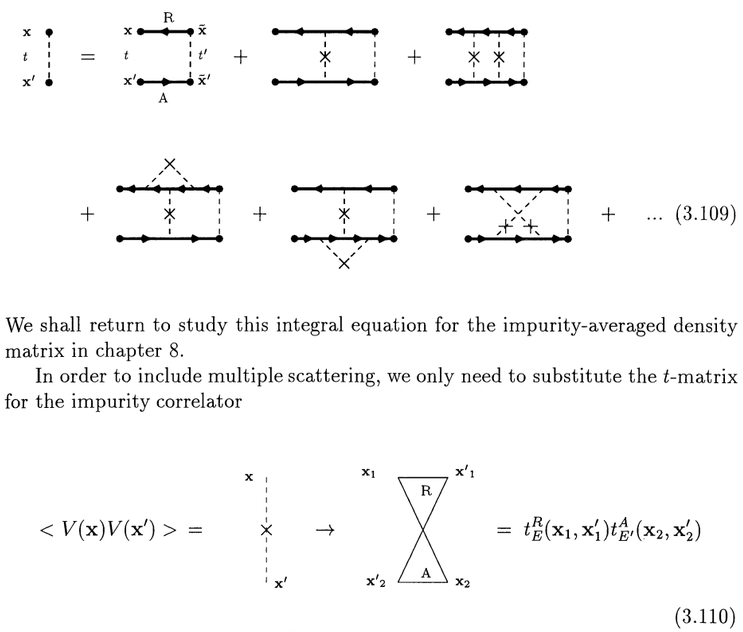
In the diagrammatic language the perturbative structure of the impurity-averaged density matrix is thus easy to grasp. We recall that on the upper line only retarded propagators appear, and on the lower line only advanced propagators.

We have drawn impurity correlators connecting upper and lower particle lines vertically, although their space-time points are different, as the presence of the

[17] For $t < t'$ the retarded and advanced labels are interchanged, and we are concerned with retrodicting the state of the particle at previous times consistent with our knowledge of the present.

retarded and advanced propagators makes the distinct topology of a diagram correspond to a unique virtual physical process. The fifth and eight diagram on the right-hand side of eq.(3.108) thus represents two physically different processes. The first contributes to the classical probability, while the other, as we shall see, represents a quantum interference process with which we shall be particularly concerned later.

Every particle line can be dressed by self-energy insertions so that the exact impurity-averaged propagator appears everywhere, and we get the skeleton diagrammatic expansion for the impurity-averaged density matrix



(3.109)

We shall return to study this integral equation for the impurity-averaged density matrix in chapter 8.

In order to include multiple scattering, we only need to substitute the t-matrix for the impurity correlator

$$= t_E^R(\mathbf{x}_1,\mathbf{x}_1')t_{E'}^A(\mathbf{x}_2,\mathbf{x}_2') \qquad (3.110)$$

here specified in terms of the Fourier transform with respect to time, and the t-matrix is carrying either an R or an A label depending on the attachment.