3 STATES, LOCALIZATION, AND CONDUCTIVITY IN ONE-DIMENSIONAL SYSTEMS

In this chapter we continue the study of one-dimensional disordered systems, examining the characteristics that are more complex than the density of states. One of the main consequences of the absence of translational invariance in disordered systems is the appearance of a macroscopically large number of localized states. This fact, which constitutes the most important difference between ordered and disordered systems, leads to a radical change in the kinetic properties of disordered systems. Such effects manifest themselves most strongly in the one-dimensional case, where even a weak random potential causes localization of all the states of the system and, as a consequence, nullifies the diffusion coefficient and dc conductivity (we are dealing with the one-body approximation, obviously). In this sense one-dimensional systems constitute a class of essentially disordered systems, so that the methods of solid-state physics prove, as a rule, to be of little use in the study of such systems.

But, as we have seen in the previous chapter, there is a very effective method of studying one-dimensional systems, a method based on the special features of one-dimensional topology. We have seen that the possibility of employing this method depends essentially on whether we can construct nonrandom equations of the Fokker–Planck type for the density of states of various quantities, which in turn depends on the property of the random potential known as the finite-Markovian property, according to which the statistical characteristics of a random potential to the right (or left) of a point are determined entirely by a finite number of parameters or, roughly speaking, by the finite radius of the interaction of the impurities that form the random

potential. As noted in Section 6.1 (see also Section 10), the energy range determines whether this property is fulfilled and what concrete form it takes. The simplest case involves a potential that is white noise uncorrelated at various points; such a case arises, as Section 6.3 has proved, in an energy range around the mean value of the potential. It is in this range that the most complete results can be obtained.

Here is a brief summary of this chapter. The wave functions of the one-dimensional Schrödinger equation with a random potential or its discrete analogs possess very special properties. It has been found that the envelope of a wave function whose logarithmic derivative is fixed at a certain point grows exponentially with the distance from the point for all realizations of the random potential. We give a proof of this statement as well as a number of quantitative results pertaining to the rate of the exponential growth (the Lyapunov exponent; see Section 10). These include several exact results and asymptotic formulas valid in the neighborhood of a stable genuine boundary of the spectrum and at sufficiently high energies (this is known as the quasi-classical region, and for it we develop a method that is a variant of the averaging method in nonlinear mechanics).

In Section 9 we explain how the property of exponential growth leads to the localization of states in one-dimensional systems of large but finite size, that is, states whose envelope decreases exponentially as one moves from the middle of the interval occupied by the system to either of its ends. This phenomenon is the simplest manifestation of localization of all the states in a one-dimensional disordered system. But to establish this property for infinite systems, where one of its forms is the discreteness of the spectrum of one-dimensional systems, requires additional arguments. The corresponding technique, which contains more mathematics than in the other parts of the book, is developed in Section 12. In Section 11 we calculate the spectral density $a(\mathbf{k}, E)$ and the average Green function in the quasi-classical region. Although both quantities are one-particle, the method employed for their calculation is a further development of the method of averaging over the fast variable formulated in Section 10.2 and proves to be the main method for calculating the low-frequency conductivity and density–density correlator at coincident energies, $p(x, E)$, in the quasi-classical region. These two quantities are found in Section 13, where we also briefly discuss the kinetics of one-dimensional disordered systems and review some further results.

9 LOCALIZATION OF STATES IN LARGE FINITE SYSTEMS

The approach developed below to localization in one-dimensional systems of large but finite length was first formulated by Mott and Twose, 1961 (see also Mott, 1967; Mott and Davis, 1979) and then analyzed by Borland (1963), and Halperin (1967).

9.1 Construction of States of One-Dimensional Finite Systems

Let us take two solutions of the Schrödinger equation, $\psi_1(x)$ and $\psi_2(x)$, that satisfy the conditions $\psi_{1,2}(\pm L) = \sin \alpha_{\pm}$ and $\psi'_{1,2}(\pm L) = \cos \alpha_{\pm}$ at the left and right ends of the interval $(-L, L)$ occupied by the system considered. Here $\alpha_{\pm}$ are the angles that fix the boundary conditions at the respective ends of the interval, so that for all the states of the system we have the following relations:

$$\frac{\psi'(L)}{\psi(L)} = \cot \alpha_+, \qquad \frac{\psi'(-L)}{\psi(-L)} = \cot \alpha_-.$$

We match these solutions at a point x_0 that lies far from both ends of the interval. Then the matching condition $\psi'_1(x_0)/\psi_1(x_0) = \psi'_2(x_0)/\psi_2(x_0)$ determines the values of E that are the energy levels for our system. (The reader can easily verify that these values do not depend on the choice of x_0.) The states that emerge from applying this procedure coincide with $\psi_2(x)$ and $\psi_1(x)$ to the left and right of point x_0. In this way we can derive information about the states of the system by studying the properties of the solutions $\psi_{1,2}(x)$.

Since the system is spatially homogeneous, the concrete position of the point is insignificant, so we assume that it coincides with the origin of coordinates. We are interested primarily in the growth and decrease of the functions $\psi_{1,2}(x)$. This implies that it is natural to introduce the function $r^2(x) = \psi^2(x) + \psi'^2(x)$, which vanishes nowhere and together with the phase $\alpha(x)$ introduced in Section 6 forms a polar coordinate system in the (ψ, ψ') "phase" plane, so that

$$\psi = r \sin \alpha, \qquad \psi' = r \cos \alpha.$$

For this reason $\alpha(x)$ is responsible for the oscillations of the wave function $\psi(x)$, and $r(x)$ for its growth or decrease. In this connection we recall the situation in the ordered case, where the potential $U(x)$ is a nonrandom periodic function of the coordinate. According to Bloch's theory, for each fixed E and $\alpha = \alpha(0)$ there is a limit

$$\lim_{|x| \to \infty} \frac{\ln r(x)}{|x|} = \gamma(E, \alpha). \tag{9.1}$$

This limit, as a function of the phase α for a fixed value of E, will either vanish for all α or assume two nonzero values, $+\gamma(E)$ and $-\gamma(E)$, with the latter value occurring only for one value of α, when in the solution of the equation which is a linear combination of exponentially increasing and decreasing functions the coefficient of the increasing term vanishes.

In a completely ordered crystal such solutions, which for large x behave like $e^{\pm \gamma x}$, play an insignificant role in forming the spectrum. Indeed, increas-

ing functions can have no part in this process, since the states emerging as a result of matching these solutions must be localized in the bulk of the crystal, whereas the periodicity conditions make this impossible. As for the exponentially decreasing functions $\psi(x)$, they can participate in the above-described matching process, but the resulting states are localized near the crystal's surface and hence constitute only a small fraction of the total number of states of a large sample. This implies, among other things, that the values of α corresponding (for fixed E) to exponentially decreasing solutions are rather rare.

Allow of this agrees well with the known fact that the energies corresponding to exponentially increasing and decreasing solutions lie in the forbidden bands of an infinite ordered crystal, while the energies from the allowed bands correspond to Bloch functions that are modulated plane waves.

However, if the periodicity is even slightly violated, for instance, by introducing a single impurity atom, then the described restriction is lifted and impurity levels appear in the forbidden bands. The corresponding impurity states are formed from functions that grow with the distance from the boundary of the crystal and hence are localized deep inside the crystal. This situation, rather untypical in ordered systems, shows, nevertheless, how localized states can form. It requires, first, that the limit (9.1) exist, and second, that is be positive for the values of energy we are interested in.

9.2 Exponential Growth of Wave Functions with the Logarithmic Derivative Fixed at a Point

In relation to the first requirement it turns out that a disordered system behaves essentially in the same manner as an ordered system. A special analysis has shown that for random potentials possessing the property of spatial homogeneity there exist the limits

$$\lim_{x \to \pm\infty} \frac{\ln r_{1,2}(x)}{|x|} = \gamma_{\pm}(E, \alpha, [U]) \tag{9.2}$$

for each fixed E and $U(x)$. These limits, if considered as functions of α, assume the values $\pm\gamma_{\pm}(E, [U])$, with $\gamma_{\pm}(E, [U]) = \max_{\alpha} \gamma_{\pm}(E, \alpha, [U])$, and the equality $\gamma_{\pm}(E, \alpha, [U]) = -\gamma_{\pm}(E, [U])$ holds only for one value of α.

The above statements are rigorous mathematical facts (see Oseledets, 1968; Millionshchikov, 1971), and constitute a generalization for the matrix case of the ergodic theorem, which we used in Section 2 to prove the self-averaging of specific extensive quantities. The reader will recall that according to this theorem, for each functional $f[U]$ given on the realizations of a spatially homogeneous random function $U(x)$, there exists with probability 1 a limit,

$$\lim_{L \to \infty} L^{-1} \int_0^L f[\hat{T}_x U] \, dx,$$

which for a $U(x)$ that weakens the correlations is nonrandom and equal to $\langle f \rangle$. ($\hat{T}_x$ is the shift operator introduced in Section 2.)

If we introduce the quantity

$$g(x_1, x_0; [U]) = \exp\left(\int_{x_0}^{x_1} f\left[\hat{T}_x U\right] dx\right),$$

the theorem can be formulated in the following manner: Given a random quantity $g(x_1, x_0; [U])$ on the realizations of the spatially homogeneous random function $U(x)$ and satisfying the relations

$$g(x_2, x_1; [U])g(x_1, x_0; [U]) = g(x_2, x_0; [U]),$$

$$g(x_1, x_0; [\hat{T}_a U]) = g(x_1 + a, x_0 + a; [U]),$$

(a) with probability 1 the limit

$$\lim_{L \to \infty} L^{-1} \ln g(L, 0; [U])$$

exists; (b) this limit is nonrandom and equal to

$$\lim_{L \to \infty} L^{-1} \langle \ln g(L, 0; [U]) \rangle$$

when $U(x)$ weakens correlations.

Let us now consider the one-dimensional Schrödinger equation as a system of two first-order equations for the two-component function $\psi(x) = (\psi(x), \psi'(x))$. Then

$$\psi(x) = \hat{G}(x, 0; [U])\psi(0), \qquad (9.3)$$

where $\hat{G}(x_1, x_0; [U])$ is a 2×2 matrix that is the solution of the equation

$$\frac{\partial \hat{G}}{\partial x_1} = \begin{pmatrix} 0 & 1 \\ U - E & 0 \end{pmatrix} \hat{G}, \qquad \hat{G}(x_0, x_0; [U]) = \hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad (9.4)$$

and satisfies the following conditions:

$$\hat{G}(x_2, x_1; [U])\hat{G}(x_1, x_0; [U]) = \hat{G}(x_2, x_0; [U]),$$

$$\hat{G}(x_1, x_0; [\hat{T}_a U]) = \hat{G}(x_1 + a, x_0 + a; [U]).$$

The first condition follows from Eq. (9.3), while the second follows from the

spatial homogeneity of $U(x)$. Since $r^2(x) = (\psi, \psi)$ is the square of the length of vector ψ, we can formulate (9.2) in the following manner: If a random matrix possesses the above properties, then in each realization there exist the following limits:

$$\lim_{|x| \to \infty} |x|^{-1} \ln(\hat{G}(x, 0)\psi_0, \hat{G}(x, 0)\psi_0),$$

with $\psi_0 = (\sin \alpha, \cos \alpha)$.

Clearly, this fact is the matrix analog of the first statement of the ergodic theorem in the multiplicative form formulated above. A full proof can be found in Oseledets (1968) and Millionshchikov (1971). A matrix analog of the second statement can also be stated. To this end we note that $\gamma(E, \alpha; [U])$ satisfies a condition that follows from the definition of this quantity [here and below we drop the plus–minus sign when we are speaking of two limits, $\gamma_\pm(E)$]:

$$\gamma(E, \alpha; [\hat{T}_x U]) = \gamma(E, \alpha(x); [U]),$$

where $\alpha(x)$ is the solution to Eq. (6.2) for which $\alpha(0) = \alpha$. Hence, for $\gamma(E; [U]) = \max_\alpha \gamma(E, \alpha; [U])$ we have the following formula:

$$\gamma(E; [\hat{T}_x U]) = \gamma(E; [U]),$$

that is, $\gamma(E; [U])$ is a random quantity that is invariant under shifts $\hat{T}_x$. As we explained in Section 2, this quantity is nonrandom if $U(x)$ weakens correlations. Hence, for the potentials considered here with α and U fixed, the rate $\gamma(E, \alpha; [U])$ can assume only two values that depend solely on E, namely $\pm \gamma(E)$, similarly to the ordered case. But in contrast to the latter case, where $\gamma(E)$ is zero at all energies belonging to the spectrum, in the disordered case $\gamma(E)$ is positive at any energy for all potentials with a finite radius of statistical correlations, excluding, perhaps, isolated energy values. The intuitively obvious reason for such unusual behavior of the solutions of the Schrödinger equation (in contrast with the periodic case) consists in the following. The typical realizations of a random potential with a finite correlation radius constitute highly irregular functions of the coordinates, that is, contain harmonics with a broad spectrum of periods. Naturally, one would think that for any energy there is a noticeable fraction of harmonics for which this energy lies in one of the forbidden bands. But then, as noted in Section 9.1, one of the two linearly independent solutions of the Schrödinger equation must be exponentially increasing, which means that the same is true of all solutions but one of the Schrödinger equation that have a logarithmic derivative fixed at zero.

It has been found, however, that a more coherent and formalized proof can be constructed more easily by using somewhat different reasoning (although

the above can also be developed into a proof of this type). This will be done in Section 10. Here, assuming that the positivity of $\gamma(E)$ is proven, we will briefly examine the conclusions about the structure of states of disordered one-dimensional systems (we will use the matching condition discussed in Section 9.1).

9.3 Discussion of the Approach

Let us assume that both α and E are fixed. Then, in accordance with (9.2), $\psi(x)$ behaves either like $e^{\gamma x}$ or like $e^{-\gamma x}$, depending on the realization of the random potential. If the second possibility is the case, the corresponding energy value is an eigenvalue of the Schrödinger equation on a semiinfinite interval with a boundary condition determined by one of the angles α_+ or α_-. But the set of the discrete levels is extremely mobile, that is, even small variations in the potential result in its motion. For this reason the probability that a fixed value of E falls into this set must be zero. (The reader will recall that a similar fact for an infinite one-dimensional disordered system was established in Section 4.) This means that for given α and E the corresponding solution, with probability 1, increases exponentially as we move away from the point at which the solution's logarithmic derivative is fixed.

But to study the localization of wave functions, we must fix the angle α (precisely, α_+ and α_-) and the realization of the random potential, while the energy must vary, so that by selecting an appropriate energy value we can match the solutions $\psi_1(x)$ and $\psi_2(x)$. It has been found that the facts make it possible only to state that the functions ψ_1 and ψ_2 grow exponentially not for all but only for almost all values of E. By "almost all" we mean that the exceptional set of energy values can be covered by a union of intervals whose length can be made as small as desired. The points of this set are, primarily, the surface levels already mentioned. An analysis shows, however, that one cannot rely only on the ergodic theorem to prove that the abovementioned set consists solely of such levels. For this reason we cannot exclude the possibility that some of the energies obtained in the matching process will be outside this set, and therefore nothing can be said of the behavior of the corresponding states. This possibility, however, has a rather low probability—as is clear, since, for one thing, the sets corresponding to ψ_1 and ψ_2 are determined by the behavior of the potential in the intervals (x_0, L) and $(-L, x_0)$, respectively, and so in general are largely independent.* As a result the majority of states constructed by matching ψ_1 and ψ_2 are localized in the middle of the interval $(-L, L)$ because of their exponential growth as we move away from the end points. This fact is substantiated by many numerical calculations

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*At the same time there are examples of random potentials for which these sets prove to be so interdependent that matching is impossible for all energies, and therefore the spectrum corresponding to such a potential contains no discrete levels. However, the described situation, for $\gamma > 0$, is unstable even for local perturbations of the random potential.

demonstrating the nature of the states of disordered one-dimensional systems consisting of 50 to 100 atoms.

Finally, we must note that this particular form of states of finite systems of sufficiently large length does not lead, generally speaking, to localization and the existence of bound states in macroscopically large (that is, infinite) one-dimensional systems. The point is that the region where the states are localized, which itself is small compared to the entire interval $(-L, L)$, may increase in such a manner that as a result we have a function that does not decrease sufficiently rapidly at infinity and therefore does not correspond to a bound state. This could mean that in macroscopically large systems there may be an intermediate length scale, much greater than interatomic distances but still small on the macroscopic scale, and that the exponential behavior of states occurs only within the limits of volumes of intermediate dimensions or, more precisely, on their periphery.

But irrespective of the above considerations, the described structure of states of one-dimensional systems appears quite remarkable. It is therefore desirable to establish such a structure in a class of cases as broad as possible. This, as we have just shown, is reduced to proving the positivity of the exponential growth rate $\gamma(E)$, to which proof we now turn.

10 THE EXPONENTIAL GROWTH RATE (LYAPUNOV EXPONENT)

This section is devoted to discussing the properties of the exponential growth rate (defined in Section 9) of a wave function with a logarithmic derivative fixed at a certain point. The exponent, as shown earlier, determines the behavior of the envelope of states of a one-dimensional disordered system with large but infinite length. As we will show in Sections 12, 13, 29, and 30, the Lyapunov exponent also plays an important role in studies of the spectrum and kinetic properties of finite (macroscopic) one-dimensional systems.

Before we elaborate the concrete results, a general remark is in order. In accordance with its definition given in Section 9, the exponent $\gamma(E)$ is the limiting value of the product of x^{-1} with the logarithm of the maximum eigenvalue of the matrix $\hat{G}$ of Eq. (9.3). According to the multiplicative ergodic theorem (see Oseledets, 1968; Millionshchikov, 1971), this quantity becomes nonrandom as $x \to \infty$, that is, undergoes self-averaging. As with other self-averaged quantities, here we can pose a question about the shape and scale of the fluctuations of the corresponding prelimiting quantity around the nonrandom limit $\gamma(E)$. It has been found (Tutubalin, 1968) that if in addition to the requirement that the statistical correlation radius of random parameters be finite, which ensures the positivity of $\gamma(E)$ (this will be shown below), we impose requirements concerning the smoothness of their probability distributions, the corresponding fluctuations will be Gaussian and the mean-square deviations in these distributions will become proportional to $x^{-1/2}$ as $x \to \infty$.

10.1 Positivity of the Lyapunov Exponent

We assume that the statistical characteristics of the potential are invariant under the transformation $x \to -x$. This invariance expresses the property of isotropy on the average, which, like homogeneity on the average, can be assumed natural for a disordered system. Since

$$\gamma_-(E, \alpha; [U]) = \gamma_+(E, \pi - \alpha; [\hat{I}U]),$$

where $\hat{I}$ is the inversion operator [it transforms $U(x)$ into $U(-x)$], and since, according to the assumption, all the various average quantities are invariant under substitution of $-x$ for x, it is clear that the two quantities $\gamma_\pm(E) = \max_\alpha \gamma_\pm(E, \alpha; [U])$, are equal to each other. For this reason it is sufficient to study $\gamma_+(E)$. Next, $\gamma_+(E)$ is a nonrandom quantity and therefore

$$\gamma_+(E) = \lim_{x \to \infty} \frac{\langle \ln r(x) \rangle}{x}. \tag{10.1}$$

But in view of the identity

$$\ln r^2(x) = 2\int_0^x \frac{\psi'}{\psi} \, dx' + \ln \psi^2(0) + \ln\left(1 + \frac{\psi'^2}{\psi^2}\right)$$

and the fact that the probability density of $\alpha(x) = \cot^{-1}(\psi'/\psi)$ usually stabilizes itself (see Chapter 2), we can write

$$\gamma(E) = \lim_{x \to \infty} x^{-1} \int_0^x \langle z(x') \rangle \, dx' = \langle z \rangle_{\text{st}}, \tag{10.2}$$

where $z(x) = \psi'(x)/\psi(x) = \cot \alpha(x)$. Thus, $\gamma(E)$ is the value of the logarithmic derivative of the wave function averaged over its stationary distribution. We will first prove that $\gamma(E)$ is positive for a model in which the potential is a sequence of nonoverlapping barriers of fixed shape, the distances between which are statistically independent. This means that

$$U(x) = \sum_j u(x - x_j), \tag{10.3}$$

where $u(x)$ is (due to isotropy) an even function equal to zero for $|x| \geq r_0$, and $x_{j+1} - x_j = y_j \geq 2r_0$, with the distances y_j being independent random quantities with the same probability density $f(y)$. For the sake of simplicity we assume that $u(x) \geq 0$. If $u(x) = \kappa_0 \delta(x)$, then (10.3) becomes the potential in (5.16), namely,

$$U(x) = \kappa_0 \sum_j \delta(x - x_j), \tag{10.4}$$

a potential we used when calculating the density of states.

If we write the Schrödinger equation as a system of two first-order equations in the form (9.3), and represent its solution in the form (9.3), we find that when in the interval $(x_j - r_0, x_j + r_0)$ the potential is symmetric with respect to the middle of the interval, the transfer matrix $\hat{G}$, which connects $\psi(x_j + r_0)$ and $\psi(x_j - r_0)$, has the form

$$\hat{G} = \begin{pmatrix} 2\alpha\beta + 1 & 2\beta \\ 2\alpha(\alpha\beta + 1) & 2\alpha\beta + 1 \end{pmatrix}.$$

Here $\alpha = c'/c$ and $\beta = cs$, while s and c are the solutions of the Schrödinger equation, taken at point $x_j + r_0$, which at point x_j satisfy the following conditions: $s = 0$, $s' = 1$, $c = 1$, and $c' = 0$. This implies that $z(x_j + r_0)$ and $z(x_j - r_0)$ are related via the linear–fractional relation containing two independent parameters α and β (at a fixed energy $E = \kappa^2$).

On the other hand, if for the potential we take the delta function with an amplitude κ_0 surrounded on both sides by empty intervals of length Δ, we find that in this particular case the transition matrix

$$\hat{G} = \begin{pmatrix} \cos 2\kappa\Delta - \dfrac{\kappa_0}{2\kappa}\sin 2\kappa\Delta & \dfrac{\sin 2\kappa\Delta}{\kappa} - \dfrac{\kappa_0}{\kappa^2}\sin^2\kappa\Delta \\ -\kappa\sin 2\kappa\Delta - \kappa_0\cos^2\kappa\Delta & \cos 2\kappa\Delta - \dfrac{\kappa_0}{2\kappa}\sin 2\kappa\Delta \end{pmatrix} \tag{10.5}$$

is also determined by two independent parameters, κ_0 and Δ. At a fixed energy these parameters can always be chosen in such a manner that the two matrices coincide, from which it follows, in particular, that

$$\kappa_0 = 2(s'c' + \kappa^2 sc). \tag{10.6}$$

But this means that in a study of the evolution of the logarithmic derivative $z(x)$, a random potential of the type (10.3) with an even function $u(x)$ that is nonzero only in the interval $|x| \leq r_0$ can be replaced with a potential of the type (10.4), whose parameter κ_0 and Δ (note that $x_{j+1} - x_j = y_j \geq 2\Delta$) are nonrandom functions of the energy, while the random distances $y_j - 2\Delta$ are the same as in (10.3).

Thus, we have reduced our problem to proving that the average value of $z = \psi'/\psi$ is positive for the potential (10.4) in which the probability density $f(y)$ of the random distances $y_j = x_{j+1} - x_j$ is zero for $y < 2\Delta$.

We will now derive an equation for the probability density of the random quantity $z(x)$ for such a potential. Actually, this equation in integral form appeared in Section 6. But in our case its differential form proves to be more convenient. This form arises if we make the probabilistic picture more detailed by fixing at point x not only $z(x)$ but $h(x)$ as well, where $h(x) - \Delta$ is the distance between point x and the closest leftward point where there is a delta

function. The joint stationary probability density $P(z, h)$ satisfies the following system of equations:

$$\frac{\partial}{\partial z}\left[(z^2 + E)P\right] - \frac{\partial P}{\partial h} - C(h)P = 0, \tag{10.7a}$$

$$P(z,0) = \int_0^\infty C(h)P(z, h)\, dh, \tag{10.7b}$$

$$P(z - \kappa_0, \Delta - 0) = P(z, \Delta + 0). \tag{10.7c}$$

Here

$$C(h) = \frac{f(h)}{F(h)}, \qquad F(h) = \int_h^\infty f(h')\, dh',$$

that is, $C(h(x))\, dx$ is the probability that the interval $(x, x + dx)$ contains point x_{j+1} under the condition that the point x_j closest to x_{j+1} on the left is separated by a distance $h(x) - \Delta$. Equations (10.7a)–(10.7c) have a simple probabilistic meaning and describe a stationary probabilistic evolution of the pair $(z(x), h(x))$. Indeed, if we note that

$$z' = -z^2 - E + U(x),$$

we can easily see that there are three situations arising in the transition from point x to point $x + dx$, namely: (a) the point (z, h) of our space of states at $h \neq 0$, Δ can be obtained only as a result of continuous movement from the state $(z + (z^2 + E)\, dx, h - dx)$; (b) the point $(z,0)$ is obtained as a result of a jump from state (z, h) for every h if there is a a scatterer at a distance Δ to the right of the interval $(x, x + dx)$; and (c) the point $(z, \Delta + 0)$ can always be obtained from $(z - k_0, \Delta - 0)$, since at $h = \Delta$ there is a delta function in the interval $(x, x + dx)$, and the logarithmic derivative z increases by κ_0 when we pass through this delta function. Equations (10.7a)–(10.7c) give the probabilities for these events. We can easily verify that at $\Delta = 0$ and $f(y) = \bar{r}^{-1}e^{-y/\bar{r}}$, Eqs. (10.7a)–(10.7c) lead to Eq. (6.72) for $P(\tilde{z}) = \int_0^\infty P(z, h)$ $(dz/d\tilde{z})\, dh$, and $z = \kappa\tilde{z}$. We now take the function

$$g(u, h) = \int_{-\infty}^\infty e^{iuz}P(z, h)\, dz. \tag{10.8}$$

For this function Eqs. (10.7a)–(10.7c) yield

$$-\frac{\partial^2 g}{\partial u^2} + gE + \frac{1}{iu}\left(\frac{\partial g}{\partial h} + Cg\right) = 0, \tag{10.9a}$$

$$g(u,0) = \int_0^\infty C(h)g(u, h)\, dh, \tag{10.9b}$$

$$g(u, \Delta - 0)e^{-iu\kappa_0} = g(u, \Delta + 0). \tag{10.9c}$$

Multiplying the second equation by $g^*(u, h)$, taking the imaginary part of the product, and integrating it from zero to infinity, we arrive at the following formula:

$$\text{Im } g^*(0, h)\frac{\partial g(u, h)}{\partial u}\bigg|_{u=0} = \text{Re}\int_0^\infty \frac{du}{u}\left(\frac{\partial g}{\partial h} + Cg\right)g^*. \tag{10.10}$$

But according to (10.8), $g^*(0, h)$ is simply $\int_0^\infty P(z, h)\, dz$, or the probability density of the random quantity h. Equations (10.7a)–(10.7c), if integrated over z, imply that this function is equal to $\bar{r}^{-1}F(h)$, where $\bar{r} = \int_0^\infty yf(y)\, dy$. Moreover, from (10.8) it also follows that

$$\frac{\partial g}{\partial u}\bigg|_{u=0} = i\int_0^\infty P(z, h)\, dz.$$

For this reason, after we divide (10.10) by $g^*(0, h) = \bar{r}^{-1}F(h)$ and integrate with respect to (10.2), on the left-hand side we will have, according to (10.2), exactly $\gamma(E) = \int P(z, h)z\, dz\, dh$, which means that

$$\gamma(E) = \frac{\bar{r}}{2}\int_0^\infty \frac{du}{u}\int_{2\Delta}^\infty \frac{dh}{F(h)}\left(2C(h)|g|^2 + \frac{\partial}{\partial h}|g|^2\right).$$

Eliminating the term $\partial|g|^2/\partial h$ by integrating by parts and by employing (10.9), we arrive at the following representation for $\gamma(E)$:

$$\gamma(E) = \frac{\bar{r}}{2}\int_0^\infty \frac{du}{u}\int_{2\Delta}^\infty f(h)\left|\frac{g(u, h)}{F(h)} - \int_\Delta^\infty \frac{g(u, h')}{F(h')}f(h')\, dh'\right|^2 dh,$$

which shows that if $\gamma(E) = 0$ for a certain energy value, then for all u and for $h > 2\Delta$ such that $f(h) > 0$ we have $g(u, h) = g(u)F(h)$, which leads, via (10.8), to $P(z, h) = P(z)F(h)$. But we can easily verify that this function at $\kappa_0 \neq 0$ cannot serve as a solution to Eq. (10.7) if there is an interval (of a length no less than $\pi/\sqrt{E}$) in which $f(h) > 0$. Thus, the fact that $\gamma(E) = 0$ implies that at this energy $\kappa_0(E) = 0$. On the other hand, if we consider the scattering of a wave incident from the left on a symmetric potential $u(x)$, we find that the appropriate transmission coefficient T has the form

$$T = \frac{\kappa^2}{\left(s'c' + \kappa^2 sc\right)^2 + \kappa^2}.$$

Combining this with (10.6), we find that the fact that $\kappa_0(E) = 0$ is equivalent to $T = 1$, which shows that at the given value of E the wave travels through the barrier without scattering, that is, its amplitude does not diminish. For

instance, for a rectangular barrier of width L and height U_0 these energies are $U_0 + n^2\pi^2/L^2$, $n = 0, 1, \ldots$ (see Landau and Lifshitz, 1977).

We have therefore established that for the potential (10.3), in which the probability density of distances $y - 2r_0$ is positive in a certain interval whose length is not less than $\pi/\sqrt{E}$, the exponential growth rate $\gamma(E)$ can vanish only at energies for which the transmission coefficient of a single scatterer is unity. Since these energy values are isolated points, whose number per unit energy interval is (in order of magnitude) no greater than $r_0/\sqrt{E}$ for large E, it is clear that their presence does not violate the above assertion that $\gamma(E)$ coincides with the limit (10.1) for almost all energies. Besides, these energy values do not exist if the potential of a single scatterer fluctuates however slightly; and there are potentials for which there are no such energies at all. A simple example is the frequently used potential formed by point scatterers, for which $u(x) = \kappa_0\delta(x)$. Such a case corresponds to $r_0 = \Delta = 0$ and κ_0 independent of E, which means that $\gamma(E)$ can vanish only if the set of values corresponding to the $f(y) \neq 0$ proves to be too "dispersed." Suppose, for instance, that $f(y) = \sum_n f_n \delta(x - na)$ (a similar potential was studied in Section 6.6 as a model of a disordered alloy). We will show that for this potential the values $E_n = \pi^2 n^2/a^2$, $n = 0, 1, \ldots$, for which an integral number of half waves fit between separate scatterers, lead to $\gamma(E) = 0$. Indeed, the reader can easily see that for such values of E the functions $\sin \kappa_n(x - ja)$ and $\sum_{i=1}^j \kappa_i \cos \kappa_n(x - ia)$, where $\kappa_n^2 = E_n$, $ja \leq x < (j+1)a$, and κ_i is κ_0 or 0 according as the ith site is occupied by a scatterer or not, form a pair of linearly independent solutions of the Schrödinger equation. But $\sum_{i=1}^N \kappa_i = \kappa_0 \bar{N}$, with $\bar{N}$ the number of scatterers in the interval $(0, L)$, and for $L \to \infty$ the ratio of the number of scatterers to that of sites tends to $\sum_{n=0}^\infty n f_n = c$. Hence $r(x)$ grows no faster than a linear function, which means that $\gamma(E) = 0$.

The above method of proving that the exponential growth rate $\gamma(E)$ is positive can be applied to a wide range of problems, for instance, to the case of random potentials that possess the finite-Markovian property in addition to such properties as spatial homogeneity, isotropy, and weakening of correlations. This means that the potential either is a Markov process (as in the case of the stochastic Kronig–Penney model of Section 6.4) or becomes a Markov process after adding a finite number of parameters $h_1, h_2, \ldots, h_n$ [a parameter of this type is above $h(x)$, which gives the distance from a given point to the closest point on the left with a delta function]. In view of this the statistical properties of the multicomponent random function $\mathbf{U}(x) = \{U(x); h_1(x), \ldots, h_n(x)\}$ for $x > x_0$ are uniquely determined if we know the statistical properties at point x. Analytically this is expressed by the fact that the probability density $P(x, \mathbf{U}|\mathbf{U}_0)$ of $\mathbf{U}(x)$ with $\mathbf{U}(0) = \mathbf{U}_0$ satisfies the Fokker–Planck equation

$$\frac{\partial P}{\partial x} = -\hat{\Lambda}_{\mathbf{U}} P, \qquad P(0, \mathbf{U}|\mathbf{U}_0) = \delta(\mathbf{U} - \mathbf{U}_0),$$

where $\delta(\mathbf{U} - \mathbf{U}_0)$ is the product of the delta functions of the continuous

variables and the Kronecker deltas of the discrete variables, and $\hat{\Lambda}_{\mathbf{U}}$ is an operator acting on the independent variables of $\mathbf{U}$ and not depending on x [due to the homogeneity of $\mathbf{U}(x)$]. For instance, for the potential (10.3) this operator is given by the right-hand side of (10.7), while for the model of rectangular random barriers of Section 6.4 it is given by the right-hand side of (6.33). But if the probability densities of the lengths of wells and barriers [$f_s(y)$, $s = 0, 1$] are arbitrary in this model, then, generalizing the reasoning that led to (6.33) and (10.7), we find that in the present case

$$\hat{\Lambda}_{\mathbf{U}} P = \frac{\partial P(s, h)}{\partial h} + C_s(h) P(s, h)$$

$$+ \delta(h) \int_0^\infty C_{1-s}(h') P(1-s, h')\, dh',$$

with

$$C_s(h) = \frac{f_s(h)}{\int_h^\infty f_s(h')\, dh'},$$

and $h(x)$ the distance from point x to the closest jump in the potential $\mathbf{U}(x)$ to the left.

In view of the same property of homogeneity, the probability density of $\mathbf{U}(x)$ is independent of x and is the solution of the stationary equation

$$\hat{\Lambda}_{\mathbf{U}} P = 0.$$

The condition of correlation weakening means that this equation has no other solutions that are strictly positive integrable functions of $\mathbf{U}$. (The reader interested in the relation between weakening of correlations—that is, ergodicity—of a random function and the uniqueness of the solution to this equation can refer to Doob, 1953.)

Let us now consider the Hilbert space $\mathscr{L}_{\mathbf{U}}$ of functions of $\mathbf{U}$ such that

$$\int |f(\mathbf{U})|^2 \frac{d\mathbf{U}}{P(\mathbf{U})} < \infty$$

(integration with respect to $\mathbf{U}$ can include summation over discrete indices). In this space we define a family of operators $\hat{P}_x$ via the following formula:

$$(\hat{P}_x f)(\mathbf{U}) = \int P(x, \mathbf{U}|\mathbf{U}') f(\mathbf{U}')\, d\mathbf{U}'.$$

The assumption that the random function $\mathbf{U}(x)$ is isotropic on the average means that the joint probability density $P(x, \mathbf{U}, \mathbf{U}_0)$ of $\mathbf{U}(x)$ and $\mathbf{U}(0)$ is

symmetric in $\mathbf{U}$ and $\mathbf{U}_0$ for every x, and hence

$$P(x, \mathbf{U}|\mathbf{U}_0) P(\mathbf{U}_0) = P(x, \mathbf{U}_0|\mathbf{U}) P(\mathbf{U}).$$

But this equation is equivalent to the self-adjointness of $\hat{P}_x$, while the fact that $P(x, \mathbf{U}|\mathbf{U}_0)$ as a function of $\mathbf{U}$ is the probability density implies that the norm of this operator does not exceed unity. Indeed, since $P(x, \mathbf{U}|\mathbf{U}_0) \geq 0$ and

$$\int P(x, \mathbf{U}|\mathbf{U}_0) \, d\mathbf{U} = 1$$

for all x and $\mathbf{U}_0$, and in view of the Schwarz inequality, we find that

$$(\hat{P}_x f, f) = \int \frac{P(x, \mathbf{U}|\mathbf{U}')}{P(\mathbf{U})} f(\mathbf{U}) f^*(\mathbf{U}') \, d\mathbf{U} \, d\mathbf{U}'$$

$$= \int \left[\frac{P(x, \mathbf{U}|\mathbf{U}') P(x, \mathbf{U}'|\mathbf{U})}{P(\mathbf{U}) P(\mathbf{U}')} \right]^{1/2} f(\mathbf{U}) f^*(\mathbf{U}') \, d\mathbf{U} \, d\mathbf{U}'$$

$$\leq \left(\int \frac{P(x, \mathbf{U}|\mathbf{U}')}{P(\mathbf{U}')} |f(\mathbf{U}')|^2 \, d\mathbf{U} \, d\mathbf{U}' \right)^{1/2}$$

$$\times \left(\int \frac{P(x, \mathbf{U}'|\mathbf{U})}{P(\mathbf{U})} |f(\mathbf{U})|^2 \, d\mathbf{U} \, d\mathbf{U}' \right)^{1/2}$$

$$= (f, f).$$

Since the conditional probability density $P(x, \mathbf{U}|\mathbf{U}_0)$, which is the kernel of $\hat{P}_x$, satisfies the Fokker–Planck equation, it is clear that

$$\hat{P}_x = \exp(-x\hat{\Lambda}_\mathbf{U}),$$

whence from the properties of $\hat{P}_x$ it follows that $\hat{\Lambda}_\mathbf{U}$ is positive definite in $\mathscr{L}_\mathbf{U}$.

We will now consider the multicomponent random function $(z(x), \mathbf{U}(x))$, where $z(x)$ is the logarithmic derivative of the wave function, as in many cases before, satisfying the equation

$$z' = -z^2 - E + U(x).$$

As explained in Section 6.4, the pair $(z(x), \mathbf{U}(x))$ constitutes a multicomponent Markov process if $\mathbf{U}(x)$ possesses this property, while the corresponding Fokker–Planck equation has the form

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial z} [(z^2 + E - U)P] - \hat{\Lambda}_\mathbf{U} P,$$

where the operator $\hat{\Lambda}_\mathbf{U}$ acts only on the (generally multidimensional) variable $\mathbf{U}$ in the function $P(x, z, \mathbf{U})$, which is the joint probability density of the random quantities $(z(x), \mathbf{U}(x))$.

Repeating almost exactly the above reasoning for the potential (10.3), we find that in the general case under consideration,

$$\gamma(E) = \int_0^\infty \frac{du}{u} \mathrm{Re}(g, \hat{\Lambda}_\mathbf{U} g),$$

where the function $g(u, \mathbf{U})$, defined in (10.8) and being the Fourier transform in z of the probability density $P(z, \mathbf{U})$, is considered as a vector in the Hilbert space $\mathscr{L}_\mathbf{U}$ depending on parameter u, while $P(z, \mathbf{U})$ is the stationary distribution of the $(z(x), \mathbf{U}(x))$ process.* But we have seen that our assumptions imply that $\hat{\Lambda}_\mathbf{U}$ is positive definite, $(g, \hat{\Lambda}_\mathbf{U} g) \geq 0$, where equality is possible only if for every u the function $g(u, \mathbf{U})$ is a eigenfunction of $\hat{\Lambda}_\mathbf{U}$, or a solution of the equation $\hat{\Lambda}_\mathbf{U} g = 0$. The last statement implies that $g(u, \mathbf{U}) = b(u)P(\mathbf{U})$, since the ergodicity of $\mathbf{U}(x)$ is equivalent to the assumption that the solution to this equation is unique. But such a function [more exactly its Fourier transform in u, of the form $\tilde{b}(z)P(\mathbf{U})$] cannot serve as a solution to the stationary Fokker–Planck equation for the pair $(z(x), \mathbf{U}(x))$,

$$\frac{\partial}{\partial z} [(z^2 + E - U)P] - \hat{\Lambda}_\mathbf{U} P = 0,$$

since this would mean that the function independent of U was a solution to the equation

$$\frac{\partial}{\partial z} [(z^2 + E - U)\tilde{b}] = 0,$$

which is obviously impossible.

Reasoning in a manner that conceptually and to a great extent technically resembles the ideas developed above, we can establish the positivity of the exponential growth rate of solutions of equations for one-dimensional discrete models with nearest-neighbor interaction (Pastur, 1980) and for the double-band model, whose density of states was calculated in Section 8 [see Eqs. (10.46) and (10.40) for calculations of $\gamma(E)$ for these two models].

All these proofs, as can be readily seen, are based essentially on the weak statistical dependence of values of the potential in regions that lie far apart. One would think, therefore, that what we are discussing here is true only when the potential possesses this property. But generally this is not so. As an illustration we will describe, following Aubrey and Andre (1980), the method

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*The fact that $g(u, \mathbf{U})$ belongs to $\mathscr{L}_\mathbf{U}$ for every u follows from an estimate that follows from (10.8), namely, $|g(u, \mathbf{U})| \leq P(\mathbf{U})$, in view of which $(g, g) \leq (P, P) = 1$.

of proving the positivity of $\gamma(E)$ for the one-dimensional Anderson model with a quasi-periodic potential. [The authors of this method used the potential as a model for describing the effect of an incommensurate structure on the motion of a quasi-particle. Such structures may appear in quasi-one-dimensional organic compounds in the presence of epitaxial layers and charge-density waves, as well as in galvanomagnetic phenomena in strong magnetic fields; see Azbel', 1964, Bulaevskiĭ, 1975, and Hofstadter, 1976.] The corresponding equation has the form

$$-\psi_{n+1} - \psi_{n-1} + U_n\psi_n = E\psi_n, \qquad U_n = g\cos(2\pi\alpha n + \omega),$$

where $g > 0$ is the coupling constant, α an irrational number (the characteristic of the incommensurate structure), and ω a random quantity uniformly distributed over the interval $[0, 2\pi]$. This interval plays the role of the realization space Ω of Section 2. As can easily be verified, such a potential satisfies the conditions of spatial homogeneity and correlation weakening formulated in Section 2. But in our case the second condition is satisfied in a weaker form than proposed in Section 2. Indeed, since

$$\langle U_n \rangle = \frac{g}{2\pi}\int_0^{2\pi}\cos(2\pi\alpha(n + \omega)\,d\omega = 0,$$

the correlation function B_n for this potential is

$$B_n = \langle U_0 U_n \rangle = \frac{g^2}{2\pi}\cos 2\pi\alpha n$$

and therefore does not tend to zero as $n \to \infty$. However,

$$N^{-1}\sum_{n=1}^{N} B_n \underset{N\to\infty}{\to} 0,$$

and, as can be verified, all higher irreducible correlators also possess a similar property. Nevertheless, this weaker form of the condition for correlation weakening is sufficient for the ergodic theorem (2.3) to be valid, and hence it is sufficient for the validity of all the results of Chapter 1. Moreover, the multiplicative ergodic theorem of Section 9.2 is valid in this case, too (see Oseledets, 1968). Therefore, for such almost periodic potentials both the density of states $\rho(E, g)$ and the Lyapunov exponent $\gamma(E, g)$ are well defined, with

$$\gamma(E, g) = \lim_{n\to\infty}(2n)^{-1}\ln(\psi_n^2 + \psi_{n+1}^2)$$

in the discrete case.

We will now show that for $g > 2$ the Lyapunov exponent γ is strictly positive for our case (Aubrey and Andre, 1980; Avron and Simon, 1982; Pastur and Figotin, 1984b). Since the width of the band of the unperturbed operator corresponding to $g = 0$ in the given case is equal to 4, the condition for $\gamma(E, g)$ to be positive can be formulated as a requirement that the potential's amplitude be greater than the half width of the band of the unperturbed translation-invariant part of the operator.

The proof is based on the following two relations:

$$\rho(E, g) = \frac{2}{g}\rho\left(\frac{2E}{g}, \frac{4}{g}\right), \tag{10.11a}$$

$$\gamma(E) = \int\ln|E - E'|\,\rho(E')\,dE'. \tag{10.11b}$$

Indeed, these relations imply that

$$\gamma(E, g) = \ln\frac{g}{2} + \gamma\left(\frac{2E}{g}, \frac{4}{g}\right),$$

and since by its very definition $\gamma(E, g)$ is nonnegative [see the text following (9.1)], we see that $\gamma(E, g)$ is positive for $g > 2$. Equation (10.11a) follows from the fact that the corresponding dynamical equation after a Fourier transformation has the same form if we substitute $2E/g$ for E and $4/g$ for g.

Equation (10.11b) is of a very general nature, and in fact is valid for all random potentials. It relates two important characteristics of disordered one-dimensional systems and was first established by Thouless, 1972 (see also Ishii, 1973; Thouless, 1974). Equation (10.11b) is a dispersion relation, since, according to the results of Sections 6 and 9, $-\pi\mathcal{N}(E)$ and $\gamma(E)$ are the values on the energy axis of the imaginary and real parts of the function

$$\lim_{n\to\infty} n^{-1}\ln(\psi_n + i\psi_{n+1}),$$

which is analytic for Im $E > 0$.

An alternative method of proving the validity of Eq. (10.11b) is based on the fact that the solution ψ_n as a function of E with a fixed ratio ψ_1/ψ_0 and fixed n has zeros at points E_{in}, $i = 1, 2, \ldots, n$, which are eigenvalues of the boundary problem on the finite interval $(1, n)$. Hence

$$\frac{1}{2n}\ln\psi_n^2 = \frac{1}{n}\sum_{i=1}^{n}\ln|E - E_{in}| = \int\ln|E - E'|\,\rho_n(E')\,dE',$$

from which we arrive at (10.11b) if we send n to ∞. Equation (10.11b) is also valid in the case of a general second-order difference equation (5.1), where it

acquires the form

$$\gamma(E) = -\langle \ln|H| \rangle + \int \ln|E - E'| \rho(E')\, dE',$$

while in the continuous one-dimensional case

$$\gamma(E) = \int \ln|E - E'|[\rho(E') - \rho_0(E')]\, dE',$$

where $\rho_0(E) = (2\pi\sqrt{E})^{-1}$ is the density of states for a free particle.

We therefore see that for strongly correlated potentials the localization of states, understood in the sense of the definitions given by Mott and Twose (1961) is possible even in one-dimensional systems only for large amplitudes of the potential, in contrast to the case of a potential with rapidly decaying correlations, for which localization occurs for an amplitude as small as desired.

Further discussions of the spectrum properties of one-dimensional equations with almost periodic coefficients are given in Section 12.

10.2 High-Energy Asymptotic Behavior of the Lyapunov Exponent and the Fast-Variable Averaging Method

According to Eq. (10.2),

$$\gamma(E) = \int z P(z)\, dz,$$

where $P(z)$ is the stationary distribution of the random quantity $z(x) = \psi'(x)/\psi(x)$. On the other hand, according to Eqs. (6.9) and (6.71), the number of states $\mathcal{N}(E)$ can also be expressed in terms of $P(z)$. For this reason the calculation of $\gamma(E)$ in closed form generally requires knowing the function $P(z)$. In Section 6.4, $P(z)$ was found for the stochastic Kronig–Penney model. Using this expression for $P(z)$, we can write a closed expression for $\gamma(E)$, which expresses $\gamma(E)$ in terms of integrals of elementary functions, just like (6.37) for $\mathcal{N}(E)$. But we will not write this formula here, since it is cumbersome. A simpler formula appears in the limiting case of white noise [see (6.3)]:

$$\gamma(E) = \mathcal{N}(E) \int_{-\infty}^{\infty} x\, dx \int_{-\infty}^{x} dy\, \exp[\Phi(y) - \Phi(x)]$$

$$= \mathcal{N}(E) \frac{\sqrt{\pi}}{2} \int_0^{\infty} \sqrt{y}\, \exp\left(-\frac{y^2}{12} - \frac{Ey}{D^{2/3}}\right) dy,$$

$$\Phi(x) = \frac{x^3}{3} + \frac{E}{D^{2/3}} x.$$

Combining this with (6.19) and (6.20), we find that

$$\gamma(E) \approx |E|^{1/2}, \qquad E \to -\infty, \tag{10.12a}$$

$$\gamma(E) \approx \frac{D}{4E}, \qquad E \to +\infty. \tag{10.12b}$$

Similar asymptotic formulas for $\gamma(E)$ can be written in the general case of the stochastic Kronig–Penney model:

$$\gamma(E) \approx |E|^{1/2}, \qquad E \to -\infty, \tag{10.13a}$$

$$\gamma(E) = \frac{1}{16(a_0 + a_1)} \frac{U_0}{E^2}, \qquad E \to +\infty. \tag{10.13b}$$

Equation (10.13a) has a simple meaning. Since the corresponding potential assumes only positive values, it can be ignored at $E \to -\infty$, and the solutions of the Schrödiger equation have the form $\exp(\pm|E|^{1/2}x)$, that is, $\gamma(E) = |E|^{1/2}$. Note that according to (10.12a) the same is true for the white-noise potential, which may have very large negative fluctuations.

Let us now turn to Eqs. (10.12b) and (10.13b). Both are first terms in the asymptotic series for large E for the integral*

$$\frac{1}{8E} \int_{-\infty}^{\infty} B_v(x) \cos 2\kappa x\, dx, \qquad \kappa^2 = E, \tag{10.14}$$

in which $B_v(x)$ stands for the correlation functions of the respective potentials in (6.11) and (6.42). This formula can also be obtained directly from the Schrödinger equation via perturbation-theory techniques. Indeed, let us introduce the phase and envelope of the wave function somewhat differently: $\psi(x) = \tilde{r}(x)\sin\varphi(x)$ and $\psi'(x) = \kappa\tilde{r}(x)\cos\varphi(x)$. Since $\lim_{x\to\infty} x^{-1}\ln \tilde{r}(x) = \lim_{x\to\infty} x^{-1}\ln r(x)$, we can find $\gamma(E)\, \tilde{r}(x)$ as well. The Schrödinger equation implies

$$\varphi' = \kappa - \frac{v(x)}{\kappa}\sin^2\varphi(x), \tag{10.15a}$$

$$\ln \tilde{r}(x) = \ln \tilde{r}(0) + \frac{1}{2\kappa}\int_0^x v(x')\sin 2\varphi(x')\, dx', \tag{10.15b}$$

and hence

$$\gamma(E) = \lim_{x\to\infty} \frac{1}{2\kappa x}\int_0^x \langle v(x')\sin 2\varphi(x')\rangle\, dx'. \tag{10.16}$$

*Here and in what follows we assume that the energy is reckoned from the average value of the potential.

When calculating $\gamma(E)$ by (10.16) with $v \ll \kappa^2$, we can employ for $\varphi(x)$ an expression derived from (10.15) by a perturbation-theory expansion. Retaining only terms of the first order in v/κ^2 and substituting the resulting expression into (10.16), we arrive at (10.14). Besides, we notice that for higher-order terms to be small the higher correlators of the potential must decay rapidly enough. Thus, the formula (10.14) for the asymptotic behavior of $\gamma(E)$ for large E is valid for a wide range of random potentials and not only for the cases (10.12) and (10.13). It implies, among other things, that the energy dependence of $\gamma(E)$ given by (10.12b) remains valid for an arbitrary potential at $E \ll r_c^{-2}$, with r_c the correlation radius of the potential $v(x)$, or the distance over which $B_v(x)$ decreases appreciably. Indeed, if this inequality is satisfied, then $\cos 2\kappa x$ in (10.14) can be replaced by unity, and we find that

$$\gamma(E) = \frac{1}{8E} \int_{-\infty}^{\infty} B_v(x)\, dx = \frac{D}{4E}, \qquad (10.17a)$$

$$l = \frac{1}{2\gamma(E)} = \frac{2E}{D}, \qquad (10.17b)$$

$$D = \int_0^{\infty} B_v(x)\, dx.$$

The quantity l defined by (10.17b) is known as the localization length and plays an important role in the studies of the spectral and kinetic properties of one-dimensional disordered systems. In the quasi-classical region $E \ll r_c^{-2}$, the localization length coincides, as we will see, with the mean free path calculated in the Born approximation [see (10.38)].

The aforesaid agrees with the results of Section 6.3, where it was found that, when calculating the average value of any function of $z = \psi'/\psi$ (or of the phase of the wave function) in a range of energies lying at a finite distance from the mean value $\overline{U} = \langle U(x) \rangle$ on the $D^{2/3}$ scale, the potential of a general type can be replaced by Gaussian white noise. In view of (10.2), $\gamma(E)$ is just such a quantity, and therefore we can apply the statement proved in Section 6.3, which means that the simple energy dependence (10.17a) for $\gamma(E)$ is limited to the following energy interval:

$$\langle v^2(x) \rangle r_c = D^{2/3} \ll E - \overline{U} \ll r_c^{-2}, \qquad (10.18)$$

where

$$U(x) = \overline{U} + v(x).$$

Hence, (10.14) is an interpolation formula that yields the asymptotic behavior of $\gamma(E)$ in both the energy interval given by (10.18) and the region $E - \overline{U} \gg r_c^{-2}$, but in the latter case $\gamma(E)$ depends on the smoothness of the

random potential realizations. We will discuss this problem in more detail below. Now we only note that (10.14) can be thought of as the generalization of (10.12b) corresponding to white noise if in (10.12b) we have

$$2D(E) = \int_{-\infty}^{\infty} B(x)\cos 2\kappa x\, dx, \qquad D = D(0),$$

instead of $2D = \int_{-\infty}^{\infty} B(x)\, dx$.

To get a better understanding of this (still formal) analogy between (10.12b) and (10.14), we derive them by somewhat different reasoning. This reasoning, closely resembling the derivation based on perturbation theory, is presented in a form that enables using it as well in calculating other characteristics of one-dimensional disordered systems in the high-energy region in a certain sense (see Section 11, 13, and 29).

We start from (10.2) and an equation for $\xi = \ln \tilde{r}$ obtained by finding the derivative of (10.15b) with respect to x:

$$\varphi'(x) = \kappa - \frac{v(x)}{\kappa}\sin^2\varphi(x), \qquad (10.19a)$$

$$\xi'(x) = \frac{v(x)}{2\kappa}\sin 2\varphi(x). \qquad (10.19b)$$

By generalizing the approach used in Chapter 2, we will think of (10.19a) and (10.19b) as equations that determine a dynamical system affected by random forces due to the potential $v(x)$. The coordinate x acts as the time variable in such a system. But a dynamical system influenced by random forces undergoes Brownian motion, which is described by a Fokker–Planck equation for the probability density of the dynamical variables, whose role in the given case is played by φ and ξ. The previous chapter has demonstrated that the form of this equation is determined essentially by the statistical properties of the random potential. As shown in Section 6.3, any potential with rapidly decaying correlations and

$$E - \overline{U} \sim D^{2/3}$$

can be replaced by white noise, or a Gaussian random function with a zero average and the correlator

$$B(x - x') = 2D\delta(x - x'). \qquad (10.20)$$

(This question is discussed at the end of Section 10 from a standpoint that differs somewhat from that presented in Section 6.3.)

But in a dynamical system of the type

$$X_i' = a_i(\mathbf{X}) + \sum_j b_{ij}(\mathbf{X}) v_j(x), \qquad (10.21)$$

where the random forces $v_j(x)$ are delta-correlated Gaussian functions

$$\langle v_i(x)v_j(x')\rangle = 2D_{ij}\delta(x-x'),\qquad(10.22)$$

the joint probability density $P(x,\mathbf{X})$ of the quantities $X_1(x),\ldots,X_n(x)$ satisfies the Fokker–Planck equation

$$\frac{\partial P}{\partial x} = -\sum_i \frac{\partial}{\partial X_i}(a_iP) + \sum_{ijln} D_{ij}\frac{\partial}{\partial X_l}\left[b_{li}\frac{\partial}{\partial X_n}(b_{nj}P)\right].\qquad(10.23)$$

To verify this we assume, for the sake of simplicity, that $n=1$ [the general case can be considered following the same scheme; e.g., see Klyatskin (1975)]. Therefore, we assume that the dynamical variable X satisfies the equation

$$X' = a(X) + b(X)v(x),\qquad(10.24)$$

with

$$\langle v(x)v(x')\rangle = 2D\delta(x-x').\qquad(10.25)$$

Differentiating the expression for $P(x,X) = \langle\delta(X-X(x))\rangle$ with respect to x and allowing for (10.24), we find that

$$\frac{\partial}{\partial x}\langle\delta(X-X(x))\rangle$$

$$= -\langle\delta'(X-X(x))(a+bv)\rangle$$

$$= -\frac{\partial}{\partial X}\langle\delta(X-X(x))a(X)\rangle$$

$$\quad -\frac{\partial}{\partial X}\langle\delta(X-X(x))b(X)v(x)\rangle$$

$$= -\frac{\partial}{\partial X}(aP) - \frac{\partial}{\partial X}\big[b(X)\langle\delta(X-X(x))v(x)\rangle\big].\quad(10.26)$$

For a further transformation of the second term in the last line we use the following formula, which is valid for any Gaussian function $v(x)$ with a zero average and correlator $B(x)$ and an arbitrary functional $R[v]$ of $v(x)$:

$$\langle v(x)R[v]\rangle = \int B(x-x')\left\langle\frac{\delta R[v]}{\delta v(x')}\right\rangle dx'.$$

When there are a finite number of Gaussian random quantities v_n, this

formula can be derived by simply integrating by parts, after which the general case is obtained as a result of a limiting process [for more details see, for instance, Klyatskin (1975)].

In our case we must take the solution $X(x)$ as the functional $R[v]$. The corresponding variational derivative $\delta X(x)/\delta v(x')$ at $x'>x$ is obviously zero, while at $x'<x$ it can be found by solving a linear differential equation obtained as a result of variational differentiation of (10.24). We find that

$$\left.\frac{\delta X(x)}{\delta v(x')}\right|_{x=x'} = b(X(x)),$$

which means that the last term on the right-hand side of (10.26) can be transformed, via (10.25), to

$$D\frac{\partial}{\partial X}\big(b(X)\langle\delta(X-X(x))b(X(x))\rangle\big) = D\frac{\partial}{\partial X}\left(b\frac{\partial}{\partial X}(bP)\right).$$

As a result, (10.26) takes the form of Eq. (10.23) with $n=1$.

Applying (10.22)–(10.23) to (10.19a) and (10.19b), we arrive at the following Fokker–Planck equation for the joint probability density $P(x,\varphi,\xi)$ of the phase $\varphi(x)$ and the logarithm $\xi(x)$ of the envelope of the wave function:

$$\frac{\partial P}{\partial x} = -\kappa\frac{\partial P}{\partial\varphi} + \frac{D}{\kappa^2}\left\{\frac{\partial}{\partial\varphi}\left[\sin^2\varphi\frac{\partial}{\partial\varphi}(\sin^2\varphi\times P)\right]\right.$$

$$-\frac{1}{2}\frac{\partial}{\partial\varphi}\left[\sin^2\varphi\frac{\partial}{\partial\xi}(\sin 2\varphi\times P)\right]$$

$$-\frac{1}{2}\frac{\partial}{\partial\xi}\left[\sin 2\varphi\frac{\partial}{\partial\varphi}(\sin^2\varphi\times P)\right]$$

$$\left.+\frac{1}{4}\frac{\partial}{\partial\xi}\left[\sin 2\varphi\frac{\partial}{\partial\xi}(\sin 2\varphi\times P)\right]\right\}.\quad(10.27)$$

If we integrate this equation with respect to ξ, we arrive (as expected) at the Fokker–Planck equation for the probability density $P(x,\varphi)$ of the phase $\varphi(x)$:

$$\frac{\partial P}{\partial x} = -\kappa\frac{\partial P}{\partial\varphi} + \frac{D}{\kappa^2}\frac{\partial}{\partial\varphi}\left(\sin^2\varphi\frac{\partial}{\partial\varphi}(\sin^2\varphi\times P)\right).\qquad(10.28)$$

As can easily be seen, this equation follows from Eq. (6.14) after the substitutions $z\to\varphi = \cot^{-1}(z/\kappa)$ and $P\to(P\sin^2\varphi)/\kappa$, which means that the solu-

tion to (10.28) can be obtained from (6.16) via the same substitutions. As for the more general equation (10.27), there is no way we can find its solution in closed form. But if we assume that $D^{2/3} \ll \kappa^2$, that is, consider only the high-energy part of the region examined, and take into account the fact that to find $\gamma(E)$ we need only know $\langle \xi(x) \rangle$, then an approximate solution can be found. To this end we note that, since $D^{2/3}$ is the only constant with the dimensions of energy that characterizes the random potential, the condition $D^{2/3} \ll \kappa^2$ means that in a certain sense the potential $v(x)$ is small. Then, as follows from (10.19a), $\varphi'(x)$ is large and depends weakly on x, which implies that the phase $\varphi(x)$ restricted to the interval $(0, \pi)$ is distributed almost evenly over that interval, and $P(x, \varphi, \xi)$ is practically independent of φ, that is, coincides with

$$\pi^{-1} P(x, \xi) = \pi^{-1} \int_0^\pi P(x, \varphi, \xi) \, d\varphi.$$

We can arrive at an equation for this function (which proves sufficient for finding $\langle \xi(x) \rangle$) if we average (10.27) over the interval $(0, \pi)$:

$$\frac{\partial P}{\partial x} = -\frac{D}{4\kappa^2} \frac{\partial P}{\partial \xi} + \frac{D}{8\kappa^2} \frac{\partial^2 P}{\partial \xi^2}. \tag{10.29}$$

If we multiply both sides of this equation by ξ and integrate with respect to ξ, we find that

$$\frac{d}{dx} \langle \xi(x) \rangle = \lim_{x \to \infty} \frac{\langle \xi(x) \rangle}{x} = \frac{D}{4\kappa^2}.$$

We have arrived at (10.17a). It can also be shown (see Khas'minskiĭ, 1966; Papanicolaou and Kohler, 1974, Gogolin *et al.*, 1975b) that when the correlation radius r_c and the de Broglie wavelength are the smallest parameters of dimensions of length but their ratio is arbitrary, the probability density of the phase nevertheless satisfies the Fokker–Planck equation. This equation has the same form as above, but now the effective diffusion coefficient is a function of E. This function proves to be the same as in (10.14), which means that the reasoning that led to (10.17) will yield (10.14) in the present (more general) situation. The same remains true for all subsequent results and formulas, which after substitution of $D(E)$ for D prove to be valid in a broader energy range.

Before we consider the other quantities, let us change the calculation tools somewhat. As we have already seen and will find below, the method of averaging over the "fast" variable is rather effective when calculating the characteristics of one-dimensional disordered systems in the energy range $D^{2/3} \ll E$ (but $E \ll r_c^{-2}$, with r_c the correlation radius of the random potential; for more details see Section 6.3). However, a disadvantage of this

method is the cumbersomeness of the formulas used to obtain the averaged equations. This becomes especially evident when one has to calculate the kinetic characteristics, in which case the Fokker–Planck equations are rather complicated. For this reason we will describe a method that enables arriving at results faster and more simply, based on the fact that the Fokker–Planck equations remain unaltered while the corresponding dynamical equations simplify. Since this reduction of the description is achieved at an earlier stage, the volume of calculations diminishes.

The approach also employs the condition $D^{2/3} \ll \kappa^2$, which means that in a certain sense the random potential is small.* However, not all the harmonics of the potential play the same role. To clarify this aspect, we turn to the analogy with nonlinear mechanics. In our case $E = \kappa^2$ plays the role of the frequency of the unperturbed motion. Since the perturbation is small, we can assume, just as is done in parametric-resonance theory (Bogolyubov and Mitropol'skiĭ, 1962), that the motion is influenced primarily by the Fourier harmonics whose numbers are either low ($|q| \ll q_0 \ll \kappa$) or equal to $\pm 2\kappa + q$. Therefore, the random potential can be written in a form that contains only "resonance" terms:

$$v(x) = v_1(x) + v_2(x) e^{2i\kappa x} + v_2^*(x) e^{-2i\kappa x}. \tag{10.30}$$

Here $v_1(x)$ incorporates only harmonics with momenta $|q| \ll q_0 \ll \kappa$, $v_2(x)$ only harmonics with momenta $2\kappa + q$, and $v_2^*(x)$ only harmonics with momenta $-2\kappa + q$; hence the functions $v_1(x)$, $v_2(x)$, and $v_2^*(x)$ vary over distances much greater than κ^{-1} and are statistically independent. On the other hand, q_0 must be selected so large that $v_1(x)$ and $v_2(x)$ vary considerably over distances that are much smaller than those over which the quantities of interest to us vary, such as the envelope of the wave function. These distances, as we have seen, are of the order of the localization length (10.17b), or $l = 2\kappa^2/D$. In calculating these quantities we can still assume $v_1(x)$ and $v_2(x)$ to be delta-correlated Gaussian functions, so that

$$\langle v_1(x) v_1(x') \rangle = \langle v_2(x) v_2^*(x') \rangle$$

$$= 2D\delta(x - x'), \tag{10.31}$$

while the other correlators vanish. We can say that the potential $v_1(x)$ is responsible for particle scattering accompanied by a small variation in momentum, while $v_2(x)$ and $v_2^*(x)$ are responsible for scattering accompanied by a change in momentum to almost the opposite, in other words, for forward scattering [$v_1(x)$] and for backscattering [$v_2(x)$ and $v_2^*(x)$].

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*An approach essentially close to the one discussed but differing somewhat in form was first applied to the theory of one-dimensional disordered systems by Abrikosov and Ryzhkin (1976) and is extensively described in the survey by Abrikosov and Ryzhkin (1978).

As we will see below, the potentials $v_1(x)$, $v_2(x)$, and $v_2^*(x)$ play different roles. To clarify this aspect, for the time being we will denote the coefficient in the correlator v_1 by D_1 and in v_2 by D_2, and then put $D_1 = D_2$ in the final result. We now introduce the reduced phase $\vartheta(x) = \varphi(x) - \kappa x$, which is a slowly varying quantity [just as $\xi(x)$ is]. Then, retaining in Eqs. (10.19a) and (10.19b) only resonance terms, we arrive at the following system of equations for ϑ and ξ:

$$\vartheta' = -\frac{v_1}{2\kappa} + f(\vartheta), \tag{10.32a}$$

$$\xi' = -\tfrac{1}{2}f'(\vartheta), \tag{10.32b}$$

with

$$f(\vartheta) = \frac{1}{4\kappa}\left(v_2^* e^{2i\vartheta} + v_2 e^{-2i\vartheta}\right). \tag{10.33}$$

Let us see how these equations lead to the formula (10.17) for the growth rate of the wave function. To this end, in accordance with (10.22)–(10.23), we construct the Fokker–Planck equation corresponding to the system (10.19a) and (10.19b):

$$\frac{\partial P}{\partial x} = \frac{2D_1 + D_2}{8\kappa^2}\frac{\partial^2 P}{\partial \vartheta^2} + \frac{D_2}{4\kappa^2}\frac{\partial P}{\partial \xi} + \frac{D_2}{8\kappa^2}\frac{\partial^2 P}{\partial \xi^2}. \tag{10.34}$$

Since to calculate $\langle \xi(x)\rangle$, which determines the growth rate, we need know only the probability density of the random quantity $\xi(x)$, Eq. (10.34) can be integrated with respect to ϑ. In the end we arrive at a closed equation containing only ξ and coinciding with (10.29).

Note that we derive the equations for the probability densities of the quantities of interest to us by integrating the more complex Fokker–Planck equations, whose construction via (10.22)–(10.23) becomes somewhat cumbersome because of the three random functions v_1, v_2, and v_2^* on the right-hand sides of the dynamical equations (10.32a) and (10.32b). To avoid this complication of the computation procedure, we reduce our description, that is, exclude the fast variable at the dynamical level from the start. To this end we bear in mind that the Fokker–Planck equation is linear in D, that is, is valid up to second-order terms in the random potential, which means that we must write the dynamical equations to the same accuracy. With this in mind, we find the solution to Eq. (10.32a) in the first order in v. Substituting the result into Eq. (10.32b), we obtain

$$\frac{d\xi}{dx} = -\frac{1}{2\kappa}\mathrm{Im}\left(v_2 e^{-2i\vartheta_0}\right)$$

$$-\frac{1}{2\kappa^2}\mathrm{Re}\left[v_2(x)\int_0^x\left(v_1(t) - \frac{v_2(t)e^{-2i\vartheta_0} + v_2^*(t)e^{2i\vartheta_0}}{2}\right)dt\right], \tag{10.35}$$

where ϑ_0 is the initial value of the reduced phase, which is independent of the values of the potential in the interval $(0, x)$. To within the accuracy we are dealing with here, the second term on the right-hand side can be replaced by its average value $D_2/4\kappa^2$, so that the dynamical equation for $\xi(x)$ takes the form

$$\xi' = \frac{D_2}{4\kappa^2} + \frac{i}{4\kappa}(\tilde{v}_2 - \tilde{v}_2^*), \tag{10.36}$$

where $\tilde{v}_2(x) = v_2(x)e^{-2i\vartheta_0}$ is a delta-correlated Gaussian function [just as $v_2(x)$ is]. For this reason $\mathrm{Im}\,\tilde{v}_2$ is a real function of the same type, and by employing (10.22)–(10.23) we arrive at the appropriate Fokker–Planck equation,

$$\frac{\partial P}{\partial x} = -\frac{D_2}{4\kappa^2}\frac{\partial P}{\partial \xi} + \frac{D_2}{8\kappa^2}\frac{\partial^2 P}{\partial \xi^2},$$

which obviously coincides with (10.29).

Returning to (10.14), we note that this formula becomes especially transparent when the potential is generated by independent scatterers, that is, in the case (10.3). Here

$$B_v(x) = \bar{r}^{-1}\int_{-\infty}^{\infty}u(x + y)u(y)\,dy, \tag{10.37}$$

and therefore (10.14) can be rewritten as follows:*

$$\gamma(E) = \frac{1}{2\bar{r}}\left|\frac{\tilde{u}(2\kappa)}{2\kappa}\right|^2, \tag{10.38}$$

where $\tilde{u}(\kappa)$ is the Fourier transform of $u(x)$, and $\bar{r}$ is the average distance between the scatterers.

We can easily show that for high energies (when the Born approximation is valid) $|\tilde{u}(2\kappa)/2\kappa|^2$ coincides with the reflection coefficient R of the potential $u(x)$. Instead of (10.38) we then have

$$\gamma(E) = \frac{1}{2\bar{r}}R(E),$$

a formula that also follows from simple considerations. According to the

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*For this model with $u(x) = \kappa_0\delta(x)$ (see Section 6) Borland (1963) found higher-order terms in the expansion in inverse powers of E [cf. (10.62a)]:

$$\gamma(E) = \frac{\kappa_0^2}{8E\bar{r}}\left(1 - \frac{\kappa_0^2}{8E}\right) + o(E^{-2}).$$