the transition process, the resulting expression being averaged over the various realizations. Within a factor of the order of unity we have

$$\sigma(\omega) \propto e^2\omega^2\rho^2(E_{\mathrm{F}}) \int |\mathbf{d}_r|^2 r^{d-1}\,dr,$$

where $\mathbf{d}_r$ is the matrix element of the coordinate between the states localized in wells separated by a distance r, and d is the dimensionality of the space. For the lower limit in this integral we can take the quantity R_ω determined by the condition $\omega \gtrsim I(R_\omega)$, with $I(r)$ the overlap integral between these states (it is here that we use the fact of repulsion of levels with close energies, that is, the fact that the resonance wells are far apart). Assuming that each of these states falls off exponentially, that is, E_{F} lies deep in the discrete spectrum, we can take the dependence of I on position in the form $I_0\exp(-r/l)$, where l is the localization radius of the states, which means that $R_\omega \propto l\ln(I_0/\omega)$. For this reason the upper limit in the integral may be taken as $R_\omega + l$. At the same time the matrix element of the coordinate between states collectivized at the resonance wells is of the order of $|\mathbf{d}_r| \propto r$. The result is the following formula (see Mott and Davis, 1979):

$$\sigma(\omega) \approx e^2\rho^2(E_{\mathrm{F}})l^{d+2}\omega^2\left(\ln\frac{I_0}{\omega}\right)^{d+1}. \qquad (4.25)$$

In conclusion we will find the relation between $f(E, E')$ of (4.24) and the correlation functions of statistical physics. To this end we introduce the current operator

$$\hat{\mathbf{j}}(\mathbf{r}) = \hat{\rho}(\mathbf{r})\hat{\mathbf{p}} + \hat{\mathbf{p}}\hat{\rho}(\mathbf{r})$$

(note that we use the system of units where $\hbar = 2m = 1$) and consider the current–current correlation function*

$$K_j(\mathbf{r}, t) = \left\langle \left\langle \hat{\mathrm{j}}^\alpha(\mathbf{r}, t)\hat{\mathrm{j}}^\alpha(0,0)\right\rangle_T \right\rangle, \qquad t > 0. \qquad (4.26)$$

The spectral density corresponding to this correlator is in essence $f(E, E + \omega)$, which by way of the Kubo–Greenwood formula gives the active conductivity, that is, the linear response function for current versus external electric field. As for the quantity similar to (4.17), it can be used (more exactly, in terms of its zeroth Fourier component in the coordinate at a frequency $\omega + i0$) to express, by way of (1.32), the total conductance of a system (i.e., including the reactance).

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*Functions similar to (4.17) and (4.26) appear also in a natural way in studies of superconductors with paramagnetic impurities (De Gennes, 1966). The reason is that because of the large correlation length, we are able to use the self-consistent-field method in the theory of superconductivity.

2 THE DENSITY OF STATES IN ONE-DIMENSIONAL SYSTEMS

This chapter is devoted entirely to one-dimensional models of disordered systems. Here, just as in other branches of theoretical physics, one can advance considerably in investigating one-dimensional versions of real three-dimensional problems. We will not dwell on the merits and shortcomings of one-dimensional models, but refer the reader to a comprehensive discussion of these problems given by Lieb and Mattis (1966). Here we wish to call attention exclusively to the basic reason why one-dimensional disordered systems in the one-body approximation frequently allow for an exact solution. The fact that the space is one-dimensional makes it possible to write closed dynamical equations (i.e., equations valid for every realization of the random potential) for the quantities that determine the spectral properties of the system. The structure of the equations is always such that on the assumption of a weak correlation of the random parameters it is possible to use the equations to obtain nonrandom equations (of the Fokker–Planck or Smoluchowski type) for the probability densities of the corresponding quantities. [The line of reasoning usually employed in its simplest form is similar to the Langevin approach to Brownian motion; e.g., see Isihara (1971), and Rytov (1976).] Sometimes these equations can be solved in closed form. When that is impossible, we can study the dynamics of the system (i.e., the properties of the appropriate stochastic equations) on characteristic parts of the spectrum and thus predict the structure of the equations for the probability distributions. From this we can develop approximate methods for finding the solutions of the equations.

As still another reason for our interest in one-dimensional models we cite the following from Dyson (1967):

> However, my personal reason for working on one-dimensional problems is merely that they are fun. A man grows stale if he works all the time on the insoluble, and a trip to the beautiful world of one dimension will refresh his imagination better than a dose of LSD. If Hans Bethe in his youth had not wasted his time solving the one-dimensional Heisenberg model of an antiferromagnet, I doubt whether he would have created the theory of energy production in stars any sooner.

Let us briefly enumerate the problems considered below. In Section 5 a method for obtaining the density of states is described in the simplest case of a one-dimensional discrete model. The method essentially goes back to Dyson (1953) (see also Lieb and Mattis, 1966) and is based on calculating the diagonal elements of the Green function outside the spectrum and then analytically continuing them onto the spectrum. Examples of the utilization of this method are given in the same section. The formalism developed in Section 6 takes account of the relationship, which is characteristic of the one-dimensional case, between the spectrum and the zeros of the solutions of the corresponding equations; it allows the density of states to be found directly, bypassing the analytic-continuation procedure. In the same section we give the exact solution of the problem of finding the density of states for the model of random rectangular barriers and investigate in detail the case of point scatterers in (1.7) with $u(x) = \kappa\delta(x)$. Section 7 is devoted to the methods of direct examination of the asymptotic behavior of the density of states in the neighborhood of the genuine boundaries of the spectrum. These methods are based on the oscillatory properties of the wave functions. The last section in this chapter, Section 8, deals with the problem of finding the density of states in the two-band model of a one-dimensional disordered system. The problem is reduced to investigating a system of equations of the Dirac type, with either the magnitude of the gap or the potential being random.

5 CALCULATING THE TRACE OF THE GREEN FUNCTION

5.1 The Formalism

Our basic object, as noted in the introduction above, is a Schrödinger equation with a random potential. However, we will begin our discussion with the discrete analog of the Schrödinger equation, just as the first works on disordered systems did. The discrete equation describes the one-dimensional motion of a particle in the tight-binding approximation or the harmonic vibrations of a one-dimensional chain, on the assumption that only the interaction of the nearest neighbors is retained. Thus the random operator we

are considering here has the form

$$\hat{\mathrm{H}} = \sum_j U_j |j\rangle\langle j| + \sum_{ij} H_{ij} |i\rangle\langle j|, \tag{5.1}$$

where $i, j = 0, \pm 1, \pm 2, \ldots,$

$$H_{j,j-1} = H_{j-1,j} = \begin{cases} H_j, & j > 0, \\ H_{j-1}, & j \le 0, \end{cases}$$

and U_j and H_j are mutually independent random variables. We do not mention the boundary condition for $\hat{\mathrm{H}}$, because according to (3.7) it is sufficient to consider a random operator without boundary conditions to calculate the density of states [i.e., we consider the operator defined by (5.1) for all j's].

A remark is in order. Since we are considering the discrete case, we cannot directly apply the proof of self-averaging of the specific extensive quantities given in Chapter 1. But if we assume that only the diagonal elements U_j are random quantities, we can build the respective proofs along the same line of reasoning as in Chapter 1 if we employ instead of the Wiener integral its discrete analog, the integral along the trajectory of the so-called Poisson process [see Dynkin, 1965, where a general method of constructing such integrals is described]. But if we wish to know only the density of states, we can resort to another method, namely, employ theorems first proved by Lederman (Born and Huang, 1954), which state that when m elements of an Nth-order matrix vary, the number of eigenvalues of this matrix in a specified interval changes by no more than m. This proposition enables us to prove the self-averaging of the number of states $\mathcal{N}(E)$ [and hence $\rho(E)$] for any operator (5.1) with an interaction that decreases fairly rapidly with increasing distance and on whose every diagonal the quantities are random sequences satisfying the conditions of spatial homogeneity and disappearance of correlations [e.g., see Slivnyak (1966) and Pastur (1971b, 1973), where this procedure was applied to the Schrödinger equation]. Finally, in the last and very general case the self-averaging of many functions such as the density of states, the spectral function, and the function $f(E_1, E_2)$ determining the conductivity can be established by proving this fact for all the moments of the respective function. Since these moments are integrals of the respective functions multiplied by integral powers of the energy, we can express them explicitly in terms of the matrix elements of the operator by way of formulas similar to (3.6) and (3.9) and hence can analyze them in detail (Pastur and Figotin, 1978a).

An operator (5.1) in which the U_j and H_j on the principal and secondary diagonals are identically distributed independent random variables obviously satisfies the above-stated conditions (Pastur and Figotin, 1978a). From (1.36) and (3.7) it follows that

$$\rho(E) = \frac{N}{\pi L}\operatorname{Im}\langle G_{00}(E - i0)\rangle, \tag{5.2}$$

where $G_{00}(E) = (E - \hat{\mathrm{H}})^{-1}_{00}$ is the diagonal matrix element of the Green function of the operator (5.1), and N and $L = Na$ is the number of sites and the length of the system. We will describe a method for finding $\langle G_{00}(E)\rangle$ in the region of negative values of E that are large in absolute value, a region where the spectrum of $\hat{\mathrm{H}}$ is absent. [This method was first developed by Dyson (1953).] Since $\langle G_{00}(E)\rangle$ is analytic in this region, it is uniquely defined for all complex values of E, which means it is also defined on the spectrum. Hence, to calculate $\rho(E)$ it is sufficient (at least in principle) to find $\langle G_{00}(E)\rangle$ for values of E lying to the left of the spectrum.

Let us define the self-energy $\Sigma_0(E)$ by the relationship

$$G_{00}(E) = [E - U_0 - \Sigma_0(E)]^{-1}. \tag{5.3}$$

Using the expansion of $\hat{\mathrm{G}}(E)$ in a perturbation series in H_{ij}, that is,

$$G_{ij} = \frac{\delta_{ij}}{e_j} + \frac{H_{ij}}{e_i e_j} + \sum_{n=2}^{\infty}\sum_{m_1,\ldots,m_{n-1}} \frac{H_{im_1}H_{m_1m_2}\cdots H_{m_{n-1}j}}{e_i e_{m_1} e_{m_2}\cdots e_{m_{n-1}} e_j}, \tag{5.4}$$

we obtain for $i = j = 0$ $(e_i = E - U_i)$

$$G_{00}(E) = e_0^{-1}\left(1 + \sum_{n=2}^{\infty}\sum_{m_1,\ldots,m_{n-1}} \frac{H_{0m_1}H_{m_1m_2}\cdots H_{m_{n-1}0}}{e_{m_1}e_{m_2}\cdots e_0}\right).$$

Each term in the double sum can be depicted as a closed polygon (loop) with n segments and with vertices at points $0, m_1, m_2, \ldots, m_{n-1}, 0$, with the factor $H_{m_i m_{i+1}}/e_{m_{i+1}}$ in the sum corresponding to the segment from vertex m_i to vertex m_{i+1}. We say that a loop is elementary if point 0 stands only as the initial and final vertex. We can then write

$$G_{00}(E) = e_0^{-1}\left(1 + \sum_{n=1}^{\infty}\sum_{p_1,\ldots,p_n} \mathfrak{M}_{p_1}\cdots\mathfrak{M}_{p_n}\right),$$

where p is a subscript that numbers the elementary loops, and $\mathfrak{M}_p$ the contribution of the pth loop. The last expression can easily be written as

$$G_{00}(E) = \left(e_0 - e_0\sum_p \mathfrak{M}_p\right)^{-1},$$

whence

$$\Sigma_0(E) = \sum_{n=2}^{\infty}\sum_{m_1,\ldots,m_{n-1}}{}' \frac{H_{0m_1}H_{m_1m_2}\cdots H_{m_{n-1}0}}{e_{m_1}e_{m_2}\cdots e_{m_{n-1}}}, \tag{5.5}$$

where the prime on the summation sign means that in every sum over m the term with $m = 0$ is absent. From this restriction and the type of interaction ($H_{ij} \neq 0$ only when $|i - j| = 1$) it follows that $m_{n-1} = m_1$, in view of which

$$\Sigma_0(E) = \sum_{m_1 \neq 0} \frac{H_{0m_1}H_{m_10}}{e_{m_1}} + \sum_{n=3}^{\infty}\sum_{m_1,\ldots,m_{n-2}}{}' \frac{H_{0m_1}\cdots H_{m_{n-2}m_1}H_{m_10}}{e_{m_1}\cdots e_{m_{n-2}}e_{m_1}}.$$

We can also write this expression in the following form:

$$\Sigma_0(E) = \sum_{m\neq 0} H_{0m}G^{(0)}_{mm}H_{m0} = H^2_{-1}G^{(0)}_{1,1} + H^2_1 G^{(0)}_{-1,-1},$$

where $G^{(0)}_{ij}$ is defined by a series (5.4) that is the same as for G_{ij}, the only difference being that in the expression for $G^{(0)}_{ij}$ none of the summation indices equals zero. If we write the same expression for $G^{(0)}_{1,1}$ and once again use the specific form of the interaction, we see that the summation is carried out only over positive values of m, that is, $m \geq 1$. This means that $G^{(0)}_{1,1}$ coincides with the 11 element of the Green function $\hat{\mathrm{G}}^{(1)}$ of the operator defined by (5.1) with $i, j \geq 1$ and represented, unlike $\hat{\mathrm{H}}$, by a semiinfinite Jacobi matrix.

Let us now consider the operator $\hat{\mathrm{H}}^{(m)}$ defined by (5.1) with $i, j \geq m$ and introduce its Green function $\hat{\mathrm{G}}^{(m)}$. For $g_m = (\hat{\mathrm{G}}^{(m)})_{mm}$ we have the following identity:

$$g_m = \left(E - U_m - H^2_{m+1}g_{m+1}\right)^{-1}, \tag{5.6}$$

which can easily be proved by way of formulas similar to (5.3)–(5.5) or by direct calculation of the appropriate matrix elements of $\hat{\mathrm{G}}^{(m)}$. Proceeding in the same way with $G^{(0)}_{-1,-1}$ and introducing some obvious notation, we obtain

$$G_{00}(E) = \left(E - U_0 - H_1^2 g_1 - H^2_{-1}g_{-1}\right)^{-1}. \tag{5.7}$$

Thus, by starting from the general equations (5.4) and (5.5) of perturbation theory, which are valid in a space with any dimensionality and an interaction of any radius, and employing the specific form of the interaction and the fact that the problem is one-dimensional, we have arrived at a reasonably simple result, whereas in the non-one-dimensional case we would have arrived at a chain of recurrence formulas of ever growing complexity, each of which would contain an infinite number of recurrent terms, instead of the compact equations (5.6) and (5.7).

From (5.6) it follows that the $u_{\pm 1} = H_{\pm 1}^2 g_{\pm 1}$ are independent of U_0 and of each other and are distributed identically. If we denote the probability densities of u and U by $P(u)$ and $Q(U)$, from (5.7) we find that

$$\langle G_{00}(E)\rangle = \int \frac{P(u)P(u')Q(U)}{E - U - u - u'}\, du\, du'\, dU. \tag{5.8}$$

But according to (5.6) the random quantities $u_m = H_m^2 g_m$ satisfy the recurrence formula

$$u_m = \frac{H_m^2}{E - U_m - u_{m+1}}, \tag{5.9}$$

which shows that u_{m+1} depends neither on U_m nor on H_m, since it is defined by the relationship derived from (5.9) by substituting $m + 1$ for m. Since the distribution functions of U_m and H_m do not depend on m, the same is true of u_m for all $m = \pm 1, \pm 2, \ldots$. Therefore, if we see the probabilities of the left- and right-hand sides of (5.9) equal, we arrive at the equation

$$P(u) = \int K_E(u, u')P(u')\, du', \tag{5.10}$$

where

$$K_E(u, u') = u^{-2} \int Q\left(E - u' - \frac{h}{u}\right) R(h)\, h\, dh,$$

and $R(h)$ is the probability density of the random quantity H_m^2. Equations (5.9) and (5.10) were first found by Dyson (1953) by averaging and a detailed analysis of the terms in the series (5.4), which defines $\hat{\mathrm{G}}(E)$.

Note that Eq. (5.10) for $P(u)$ is not, strictly speaking, well defined, since this function may prove to be highly irregular. For example, it may have delta-like singularities on a dense set of points (Lieb and Mattis, 1966; Schmidt, 1957). But if we integrate Eq. (5.10) with a smooth function $f(u)$, we arrive at an equation that is well defined:

$$\int f(u)P(u)\, du = \int f\left(\frac{h}{E - U - u}\right) P(u)Q(U)R(h)\, du\, dU\, dh.$$

The irregular behavior of $P(u)$ may generally cause the appearance of similar singularities in the density of states. This, however, does not lead to singularities in physical quantities, since the latter are usually expressed in terms of integrals of $\rho(E)$ with smooth functions. [Note that the functions with which $\rho(E)$ are integrated may also be not very smooth. Specifically, the number of states

$$\mathcal{N}(E) = \int_{-\infty}^{E} \rho(E')\, dE' = \int_{-\infty}^{\infty} \theta(E - E')\rho(E')\, dE',$$

where $\theta(E)$ is the Heaviside function, can be shown to be a continuous function of E if the conditions of spatial homogeneity and disappearance of correlations are fulfilled.] Moreover, these irregularities occur only when the random parameters in the problem take on none but discrete values, that is, when their probability densities have delta-like peaks and vanish for an arbitrary small broadening of these peaks.

By using Eqs. (5.2), (5.8), and (5.10) we can solve the problem of finding the density of states $\rho(E)$ (at least in principle). To this end, according to the above reasoning, we must solve the integral equation (5.10) for the density of states $P(u)$, which depends on E as on a parameter, compute $\langle G_{00}(E)\rangle$ by (5.8), analytically continue the function obtained into the neighborhood of the spectrum, and, finally, employ (5.2).

The right-hand side of (5.8), which defines $\langle G_{00}(E)\rangle$, depends bilinearly on $P(u)$. Similar bilinear formulas for the density of states and a number of other quantities that characterize one-dimensional disordered systems will often be used below (see Sections 11–13). We can also arrive at a formula for $\langle G_{00}(E)\rangle$ in which $P(u)$ enters linearly. To this end we take a long but finite chain whose Hamiltonian $\hat{\mathrm{H}}_N$ is defined by (5.1) with $i, j = 1, \ldots, N$ and zero boundary conditions. In accordance with (3.6) and (1.36) we have

$$\begin{aligned}\rho(E) &= \pi^{-1} \mathrm{Im} \lim_{N\to\infty} (Na)^{-1} \mathrm{Tr}\, \hat{\mathrm{G}}^{(N)}(E - i0)\\ &= (\pi a)^{-1} \mathrm{Im} \lim_{N\to\infty} N^{-1} \sum_{m=0}^{N} G_{mm}^{(N)}(E - i0),\end{aligned}$$

where $\hat{\mathrm{G}}^{(N)} = (E - \hat{\mathrm{H}}_N)^{-1}$. As above, we will assume that E lies outside the spectrum. For such E we have, by virtue of (3.6), the following formula:

$$\langle G_{00}(E)\rangle = \lim_{N\to\infty} N^{-1} \langle \mathrm{Tr}\, \hat{\mathrm{G}}^{(N)}(E)\rangle,$$

or

$$\begin{aligned}\langle G_{00}(E)\rangle &= \lim_{N\to\infty} N^{-1} \frac{d}{dE} \left\langle \mathrm{Tr} \ln\left(E - \hat{\mathrm{H}}_N\right)\right\rangle\\ &= \lim_{N\to\infty} N^{-1} \frac{d}{dE} \langle \ln D_N\rangle,\end{aligned}$$

where $D_N \equiv \det(E - \hat{\mathrm{H}}_N)$ is the determinant of the following Nth-order matrix:

$$E - \hat{\mathrm{H}}_N = \begin{pmatrix} E - U_1 & -H_1 & 0 & & & & \\ -H_1 & E - U_2 & -H_2 & & & & \\ 0 & -H_2 & E - U_3 & & & & \\ & & & \ddots & & & \\ & & & & E - U_{N-2} & -H_{N-2} & 0 \\ & & & & -H_{N-2} & E - U_{N-1} & -H_{N-1} \\ & & & & 0 & -H_{N-1} & E - U_N \end{pmatrix}.$$

Let us denote the determinant derived from D_N by deleting out the last $N - m$ rows and columns by D_m. Expanding D_{m+1} in the elements of the last row, we arrive at the recurrence formula

$$D_{m+1} = (E - U_{m+1})D_m - H_m^2 D_{m-1}.$$

This implies that the $u_m^{(N)}(E)$ defined as

$$u_m^{(N)} = \frac{H_m^2 D_{m-1}}{D_m}$$

satisfy, in turn, the recurrence formula

$$u_m^{(N)}(E) = \frac{H_m^2}{E - U_m - u_{m-1}^{(N)}(E)}, \quad m = 1, \dots, N-1, \qquad u_0^{(N)}(E) = 0, \tag{5.11}$$

and depend only on $U_1, \dots, U_m, H_1^2, \dots, H_m^2$, with the explicit form of the dependence on U_m and H_m^2 indicated by this formula.

In terms of the $u_m^{(N)}$ the determinant D_N has the form

$$D_N = \prod_{m=1}^{N} \left[E - U_m - u_{m-1}^{(N)}(E)\right],$$

whence

$$\langle G_{00}(E)\rangle = \frac{d}{dE} \lim_{N\to\infty} N^{-1} \sum_{m=1}^{N} \left\langle \ln\left[E - U_m - u_{m-1}^{(N)}(E)\right]\right\rangle. \tag{5.12}$$

Comparing (5.11) with (5.9), we see that in the limit as $N \to \infty$ the $u_m^{(N)}$ can be identified with the u_m. Hence, in view of the independence of u_{m-1} and U_m, it follows from (5.12) that

$$\langle G_{00}(E)\rangle = \frac{d}{dE} \int P(u)Q(U)\ln(E - U - u)\, dU\, du. \tag{5.13}$$

Substituting this equation into (5.2), we see that instead of $\langle G_{00}(E)\rangle = \langle (E - \hat{\mathrm{H}})_{00}^{-1}\rangle$ we can consider the function

$$\begin{aligned}\Omega(E) &= \lim_{N\to\infty} N^{-1} \left\langle \ln\det\left(E - \hat{\mathrm{H}}_N\right)\right\rangle \\ &= \int P(u)Q(U)\ln(E - U - u)\, dU\, du. \end{aligned}\tag{5.14}$$

The number of states (or the integrated density of states) $\mathcal{N}(E) = \int_{-\infty}^{E} \rho(E')\, dE'$ can be expressed in terms of the analytic continuation of $\Omega(E)$ in the same manner as $\rho(E)$ is expressed in terms of the continuation of $\langle G_{00}(E)\rangle$:

$$\mathcal{N}(E) = (\pi a)^{-1} \operatorname{Im} \Omega(E - i0). \tag{5.15}$$

It may seem that in deriving Eqs. (5.8) and (5.13) we used not only the fact that the problem is one-dimensional and the type of interaction (nearest-neighbor interaction) but also the fact that the problem is discrete. This last fact, however, in contrast to the first two conditions, is not really essential, so that the scheme expounded above can be carried over to the continuous case, that is, to the one-dimensional Schrödinger equation with a random potential. We will not dwell on this, but merely note a special case where the analogy with the discrete problem is most complete. Namely, we consider the one-dimensional Schrödinger equation in which the potential $U(x)$ has the form (1.7) with $u(x) = \kappa_0\delta(x)$, $\kappa_0 > 0$. In other words, let us consider the operator $\hat{\mathrm{H}}_L$ defined by the equation

$$-\psi'' + U(x)\psi = E\psi, \qquad U(x) = \kappa_0 \sum_j \delta(x - x_j), \tag{5.16}$$

and the boundary conditions $\psi(L) = \psi(-L) = 0$, assuming, as previously, that $E = -q^2$ lies outside the spectrum of $\hat{\mathrm{H}}_L$. Employing the specific form of $U(x)$, we can easily show that

$$\langle G(0,0;E)\rangle = \lim_{L\to\infty} (2L)^{-1}\left(\operatorname{Tr}\hat{\mathrm{G}}_L^{(0)} + \frac{d}{dE}\ln\det\left(\mathrm{I} - k_0\hat{\hat{\mathrm{G}}}_L^{(0)}\right)\right). \tag{5.17}$$

Here $\hat{\hat{\mathrm{G}}}_L^{(0)}$ is an $\overline{N}$-by-$\overline{N}$ matrix, with $\overline{N}$ the number of scatterers in the interval $(-L, L)$ and

$$\left(\hat{\hat{\mathrm{G}}}_L^{(0)}\right)_{ij} = G_L^{(0)}(x_i, x_j; E).$$

Thus the problem is again reduced to calculating a determinant. In the given case it is the determinant $D_{\overline{N}}$ of order $\overline{N}$ with elements D_{ij} of the form

$$D_{ii} = 1 + \lambda, \quad D_{i,i+k} = D_{i+k,i} = \lambda \prod_{j=1}^{k} \eta_{i+j}, \qquad 1 \le i < i + k \le \overline{N},$$

where

$$\lambda = \frac{\kappa_0}{q}, \qquad \eta_j = \exp(-qy_j), \qquad y_j = x_{j+1} - x_j.$$

If we denote by D_m the determinant derived from $D_{\overline{N}}$ by retaining only the first m rows and columns, then we arrive, as before, at a recurrence formula, which in terms of the $u_m^{(\bar{N})} = D_m/D_{m+1}$ has the form

$$u_m^{(\bar{N})}(E) = \frac{H^2(y_m)}{U(y_m) - u_{m-1}^{(\bar{N})}(E)}, \tag{5.18}$$

which is similar to (5.9). Here $H(y_m) = \eta_m^{-1}$ and $U(y_m) = 1 - \lambda + (1 + \lambda)\eta_m^{-2}$. The $u_m^{(\bar{N})}(E)$ depend on $y_1, \dots, y_m$, with the explicit form of the dependence on y_m given by (5.18). The sought-for determinant $D_{\bar{N}}$ can be simply expressed in terms of the $u_m^{(\bar{N})}$ thus:

$$D_{\bar{N}} = \frac{1 + \lambda}{\prod_{m=1}^{\bar{N}-1} u_m^{(\bar{N})}}. \tag{5.19}$$

As a result we obtain the following equation for $\langle G(0,0;E)\rangle$ from (5.17):

$$\langle G(0,0;E)\rangle = -\frac{1}{2q} - \frac{1}{\bar{r}}\frac{d}{dE}\int P(u) \ln u \, du,$$

where the probability density $P(u)$ of u satisfies, according to (5.18), the following equation:

$$P(u) = \int P(u') f(y)\, \delta\left(u - \frac{H^2(y)}{U(y) - u'}\right) du'\, dy, \tag{5.20}$$

in which $f(y)$ is the probability density of the distance between two neighboring scatterers, and $\bar{r} = 2L/\bar{N}$ the mean distance between them.

5.2 The Dyson Model

Following Dyson (1953), we start with a particular case where the diagonal elements U_j in (5.1) are zero while the off-diagonal elements H_j (the resonance integrals) are uncorrelated random quantities distributed according to the same law (the so-called Γ-distribution)

$$P_n(h) = \frac{n^n}{\Gamma(n)} h^{n-1} e^{-nh}, \qquad h = \frac{H^2}{H_0^2}. \tag{5.21}$$

The average value of H^2/H_0^2 corresponding to this probability density tends to unity as $n \to \infty$, while its root-mean-square is equal to $n^{-1/2}$. For this reason the number n can be taken as the measure of disorder in the system. In the limit as $n \to \infty$ we arrive at a completely ordered system in which all H_j are equal to H_0.

The distribution (5.21) has no physical model that is at all realistic. We mention it only because of its simplicity and the possibility of obtaining a closed expression for the density of states. The limiting form of the expression (5.11) with $N \to \infty$ is in the present case

$$u_m = \frac{H_m^2}{E - u_{m-1}}. \tag{5.22}$$

The random quantities H_m corresponding to distribution (5.21) can assume any values, which means that the spectrum of the system in question occupies the entire energy axis. Since in the formulas derived in Section 5.1 the energy E cannot belong to the spectrum, we must assume that it is complex-valued. It is convenient to put E purely imaginary, or $E = i\zeta$, and introduce the variable $\xi_m = -u_m/E$ instead of u_m. As a result, instead of (5.22) we have

$$\xi_m = \frac{\lambda h_m}{1 + \xi_{m-1}}, \qquad m = 1, 2, \dots, \tag{5.23}$$

where $\lambda = \zeta^{-2} H_0^2$ and $h_m = H_m^2/H_0^2$. Instead of (5.14) we have

$$\Omega(i\zeta) = \ln i\zeta + \int_0^\infty \ln(1 + \xi)\, P(\xi)\, d\xi, \tag{5.24}$$

where $P(\xi)$ is the stationary probability density of ξ, for which we have the following equation similar to (5.20):

$$P(\xi) = \int_0^\infty P_n\left(\frac{\xi(1 + \xi')}{\lambda}\right) P(\xi') \frac{1 + \xi'}{\lambda}\, d\xi$$

[this equation follows from (5.23)]. Direct verification shows that when P_n is given by (5.21), the solution of the above equation is given by the function

$$P(\xi) = M_n^{-1}(\lambda)\xi^{n-1}(1 + \xi)^{-n} \exp\left(-\frac{n\xi}{\lambda}\right),$$

where $M_n(\lambda)$ is a normalization constant:

$$M_n(\lambda) = \int_0^\infty \xi^{n-1}(1 + \xi)^{-n} \exp\left(-\frac{n\xi}{\lambda}\right) d\xi.$$

Now instead of (5.24) we have

$$\Omega(i\zeta) = \ln i\zeta + \frac{L_n(\lambda)}{M_n(\lambda)},$$

with

$$L_n(\lambda) = \int_0^\infty \xi^{n-1}(1 + \xi)^{-n} \ln(1 + \xi) \exp\left(-\frac{n\xi}{\lambda}\right) d\xi.$$

On the other hand, the definition (5.14) of the function $\Omega(E)$ implies that

$$\Omega(E) = \int_0^\infty \ln(E^2 - E'^2)\,\rho(E')\,dE,$$

since $\rho(E)$ in the system we are considering here ($U_j = 0$) is an even function of energy. [This follows from the relation $\hat{\mathrm{D}}\hat{\mathrm{H}}_N\hat{\mathrm{D}}^{-1} = -\hat{\mathrm{H}}_N$, where $\hat{\mathrm{D}}$ is a diagonal matrix with elements $d_i = \prod_{m=1}^{i}(-H_m)$. It also means that the off-diagonal elements H_m can always be taken positive.] This formula is similar to the original one, except that for the energy variable we must take E^2. For this reason, the analytic continuation of (5.15) in E^2 into the range of negative values yields

$$\tilde{\mathcal{N}}(E) = \int_0^E \rho(E')\,dE'$$

instead of the integrated density of states

$$\mathcal{N}(E) = \int_{-\infty}^{E} \rho(E')\,dE',$$

which, however, is sufficient for calculating $\rho(E)$ in view of the evenness of the latter.

Performing analytic continuation of $M_n(\lambda)$ and $L_n(\lambda)$ into the range of negative values of λ [following Dyson (1953); see also Smith (1970)] and using (5.15), we arrive at an explicit expression for $\tilde{\mathcal{N}}(E)$, which in the simplest case with $n = 1$ assumes the form

$$a\tilde{\mathcal{N}}(E) = \frac{G_1^2 + G_2 e^{-\varepsilon} + (\pi^2/6)e^{-2\varepsilon}}{\left[G_1^2 + (\mathbb{C} - \ln\varepsilon)e^{-\varepsilon}\right]^2 + \pi^2 e^{-2\varepsilon}},$$

where $\varepsilon^{-1} = E^2/H_0^2$, $\mathbb{C}$ is Euler's constant, and

$$G_1 = \sum_{m=1}^{\infty} \frac{(-\varepsilon)^m s_m}{m!}, \qquad G_2 = \sum_{m=1}^{\infty} \frac{(-\varepsilon)^m (s_m^2 + t_m)}{m!},$$

$$s_m = \sum_{p=1}^{m} p^{-1}, \qquad t_m = \sum_{p=1}^{m} p^{-2}.$$

This formula implies, in particular, that $\tilde{\mathcal{N}}(E)$ for small E has the form

$$\tilde{\mathcal{N}}(E) = \frac{\pi^2}{6a}\ln^{-2}\varepsilon, \qquad E \to 0,$$

according to which the density of states for small E, that is, in the middle of the band, has a singularity:

$$\rho(E) \approx \frac{\pi^2}{12a}\left|E\ln^3\left(\frac{E}{H_0}\right)\right|^{-1}, \qquad E \to 0. \tag{5.25}$$

This singularity is caused by the fact that each of the quantities H_m distributed according to the law (5.21) can assume arbitrarily small values. To better understand this statement let us take the simple case where the H_m can assume zero values with a nonzero probability c. Then the initial matrix $\hat{\mathrm{H}}_N$ splits into independent blocks of random sizes, and in the middle of the band there appears a singularity $c(2a)^{-1}\,\delta(E)$, since each odd-order block will have a level at zero (all the blocks, as noted earlier, have spectra that are symmetric with respect to zero).

As for the singularities in the density of states that in an ordered system were at the band edges [$\rho(E) = (2\pi a H_0)^{-1}(1 - E^2/4H_0^2)^{-1/2}$ if all H_m are equal to H_0], they all disappear in the disordered case. The behavior of $\rho(E)$ at small disorder ($n \to \infty$) was analyzed in detail by Smith, 1970, who established the way in which these singularities smear out, as well as the asymptotic behavior of $\rho(E)$ as $n \to \infty$ in the entire energy range (see also Section 5.3).

If the Fermi level occurs in the middle of the band and the density of states has singularities at this point, the result is singularities in thermodynamic quantities. For instance, the zero-field magnetic susceptibility of a disordered system with $T \to 0$ acquires the form

$$\chi(T) \sim \int_{-\infty}^{\infty} \frac{\rho(E_{\mathrm{F}} + 2Tx)}{\cosh^2 x}\,dx.$$

Hence at $E_{\mathrm{F}} = 0$ and $T \to 0$ we have (Bulaevskiĭ *et al.*, 1972)

$$\chi(T) \sim T^{-1}\ln^{-2}\left(\frac{T}{H_0}\right).$$

A similar singularity appears in the X–Y model, which (see Lieb and Mattis, 1966) is equivalent to a gas of free fermions with energies that are eigenvalues of the matrix $\hat{\mathrm{H}}_N$ considered; this matrix plays the role of the exchange-integral matrix in the X–Y model. The model was used by Bulaevskiĭ *et al.* (1972), to describe the magnetic properties of linear chains of high-conductivity complexes of TCNQ.

The thermodynamic properties of disordered X–Y chains were also studied by Smith (1970), who showed that such systems have no phase transitions in the field at $T = 0$, in contrast to ordered systems.

5.3 The Asymptotic Behavior of the Density of States Near the Genuine Boundary and in the Neighborhood of the Edge of the Initial Spectrum in the Model of Point Scatterers

We start by evaluating the asymptotic behavior of the density of states in the model (1.7) with $u(x) = \kappa_0 \delta(x)$, $\kappa_0 > 0$ (repulsive point scatterers), near the genuine boundary of the spectrum $E_g = 0$, that is, for $q \ll \kappa_0$. To allow for this condition, we rewrite (5.18) in such a way that it allows for the explicit dependence on y_j:

$$u_j^{-1}(E) = \frac{\kappa_0}{q}\left[1 - \exp(-2qy_j)\right] + \left[1 + \exp(-2qy_j)\right]$$

$$-\exp(-2qy_j)\, u_{j-1}(E). \tag{5.26}$$

Since the first term on the right-hand side contains the large parameter κ_0/q, we can expect that the u_j determined by this relationship will be small. But if we want this to be true, we must require that $\kappa_0 \bar{r} \gg 1$ [since κ_0/q enters (5.26) together with the factor $1 - \exp(-2qy_j)$], which means that the scatterer concentration $c = (\kappa_0 \bar{r})^{-1}$ must be low (small disorder). Then in (5.26) we can retain only the first term, and hence the $u_j(E)$ are expressed in this approximation in terms of the distance y_j between neighboring scatterers:

$$u_j^{-1}(E) \approx \frac{\kappa_0}{q}\left[1 - \exp(-2qy_j)\right].$$

So from (5.19) we obtain

$$\langle G(0,0;E)\rangle = -\frac{1}{2q} + \frac{1}{2q^2\bar{r}}$$

$$-\frac{1}{q\bar{r}} \sum_{n=1}^{\infty} \int_0^{\infty} y f(y) \exp(-2qny)\, dy,$$

where $f(y)$ is the probability density of the distances y between neighboring scatterers. This expression can be continued onto the spectrum by substituting $i\kappa$ for q, with $\kappa^2 = E > 0$. After simple transformations we obtain

$$\rho(E) = \frac{\pi}{2\kappa^3 \bar{r}} \sum_{n=1}^{\infty} n f\left(\frac{n\pi}{\kappa}\right), \tag{5.27}$$

which is valid for an arbitrary distribution $f(y)$ with a first moment.

Two regions can be singled out in the applicability range $\kappa \ll \kappa_0$ of (5.27) (the concentration is low: $\kappa_0 \bar{r} \gg 1$), namely $\kappa \ll \bar{r}^{-1}$ and $\bar{r}^{-1} \ll \kappa \ll \kappa_0$. If the function $f(y)$ falls off no more slowly than the exponential function, then in the first region ($\kappa \ll \bar{r}^{-1}$) the main contribution to the density of states is provided by the first term on the right-hand side of (5.27) and

$$\rho(E) \approx \frac{\pi}{2\kappa^3 \bar{r}} f\left(\frac{\pi}{\kappa}\right), \qquad E \ll \bar{r}^{-2}. \tag{5.28}$$

Here the density of states becomes exponentially small as $\kappa \to 0$, because of the fluctuation nature of the states near the boundary $E_g = 0$. We will return to this question in Section 7 and in Chapter 4. But if $f(y)$ drops off like a power function, $f(y) \propto \bar{r}^{-1}(\bar{r}/y)^{\alpha}$, $\alpha > 2$, we must take into account all terms, which yields

$$\rho(E) \approx \frac{1}{2\pi\kappa}\left(\frac{\kappa\bar{r}}{\pi}\right)^{\alpha-2} \zeta(\alpha - 1), \tag{5.29}$$

with $\zeta(x)$ the Riemann zeta function. In the second region ($\bar{r}^{-1} \ll \kappa \ll \kappa_0$), summation in (5.27) can be replaced by integration. Proceeding with this and allowing for the fact that $\int_0^{\infty} y f(y)\, dy = \bar{r}$, we find that

$$\rho(E) \approx (2\pi\kappa)^{-1}, \qquad \bar{r}^{-2} \ll E \ll \kappa_0^2. \tag{5.30}$$

In the region where $\kappa \approx \bar{r}^{-1}$ the two formulas (5.28) and (5.30) can be matched, which yields $\rho \propto \bar{r}$.

In the special case of the Poisson distribution $f(y) = \bar{r}^{-1}\exp(-y/\bar{r})$, (5.27) directly yields (Bychkov and Dykhne, 1966a)

$$\frac{\kappa_0}{2}\rho(E) \approx \frac{2\pi c^2}{\varepsilon^{3/2}} \frac{\exp(-2\pi c/\sqrt{\varepsilon})}{\left[1 - \exp(-2\pi c/\sqrt{\varepsilon})\right]^2}, \qquad \varepsilon = \frac{4E}{\kappa_0^2}. \tag{5.31}$$

In the limiting cases we have

$$\frac{\kappa_0}{2}\rho(E) \approx \begin{cases} \dfrac{2\pi c^2}{\varepsilon^{3/2}} \exp\left(-\dfrac{2\pi c}{\varepsilon^{1/2}}\right), & \varepsilon \ll c^2, \\[2ex] \dfrac{1}{2\pi\varepsilon^{1/2}}, & c^2 \ll \varepsilon \ll 1. \end{cases} \tag{5.32}$$

Thus, a small disorder ($c \ll 1$) leads to a situation in which the density of states $\rho(\varepsilon)$ (the solid curve in Figure 2) differs from the density of states $\rho_0(\varepsilon)$ of the ideal system (the dashed curve) only in a narrow neighborhood $\Delta\varepsilon \approx c^2$ of the true fluctuation boundary of the spectrum.

Mints, 1966, arrived at results that practically coincide with (5.32). They were obtained in the tight-binding approximation (5.1) with off-diagonal disorder, when $U_j = 0$ and the $h_j = H_j^2$ are independent and distributed

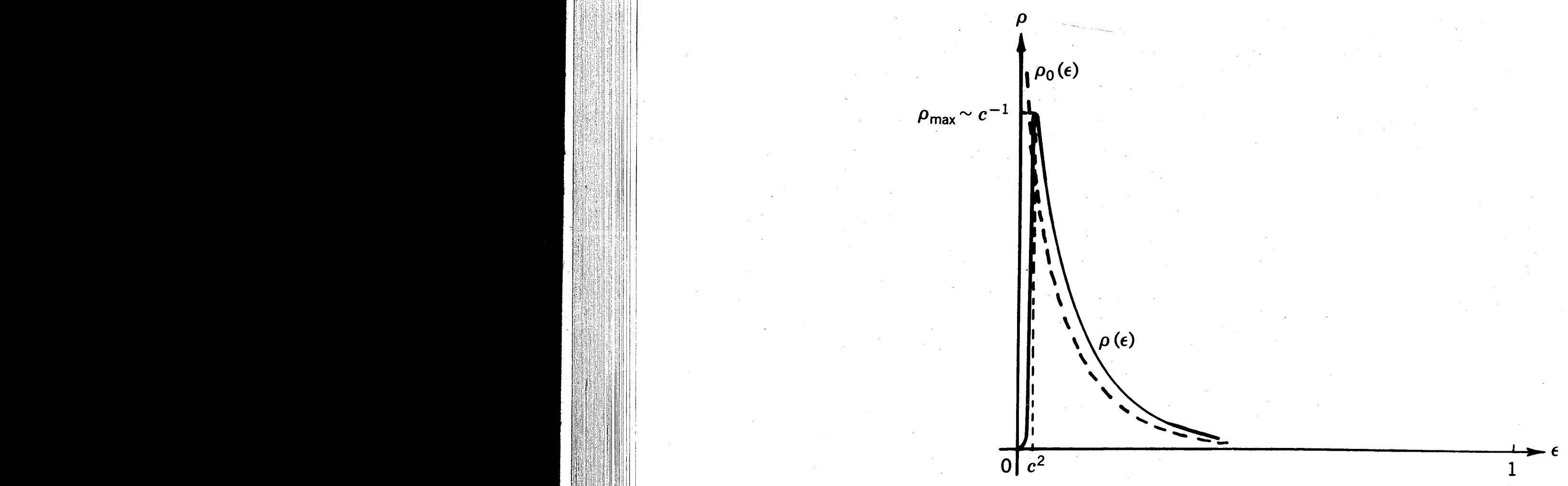

Figure 2

similarly with a probability density $P(h)$, $0 < h < h_0 < \infty$. Mints also considered one case of interdependent H_j, which actually formed a Markov chain.

For the model with attractive scatterers ($\kappa_0 < 0$) the boundary of the spectrum is at $-\infty$, but for low concentrations $c = (|\kappa_0|\bar{r})^{-1} \ll 1$ it is possible to examine the behavior of the density of states in the neighborhood of the boundary of the initial spectrum $E = 0$. In this case the recurrence formula (5.26) takes the form

$$u_j^{-1} = -\frac{\kappa_0}{2iq}\left[1 - \exp(2iqy_j)\right] + \left[1 + \exp(2iqy_j)\right] - \exp(2iqy_j)\, u_{j-1}, \quad (5.33)$$

with q in the first quadrant of the complex q-plane. Real values of q correspond to positive energies, and purely imaginary values of q to negative energies. As in the case above, for $|q| \ll |\kappa_0|$, $|\kappa_0|\bar{r} \gg 1$, we can neglect the third term in (5.33). As a result the recurrence formulas are recoupled and we have

$$u_j^{-1} = u^{-1}(y_j) = -\frac{\kappa_0}{2iq}\left[1 - \exp(2iqy_j)\right] + \left[1 + \exp(2iqy_j)\right].$$

(The second term, however, must be taken into account, since it is this term that leads to a nonzero density of states in the region $E < 0$.)

In this approximation,

$$\rho(E) = \frac{1}{2\pi}\operatorname{Re}\frac{1}{q} + \frac{1}{2\pi\bar{r}}\operatorname{Im}\frac{1}{q}\frac{d}{dq}\int_0^\infty f(y)\ln u(y)\, dy.$$

Continuing this equation analytically onto the spectrum ($\operatorname{Im} q \to +0$ or $\operatorname{Re} q \to +0$) and evaluating the integral on the right-hand side, we arrive at

the following expression (Gredeskul and Trifonov, 1976) for the density of states in the energy range $|\varepsilon| \ll 1$:

$$\frac{|\kappa_0|\rho(E)}{2} \approx \begin{cases} \frac{4}{3}c^2\left(\dfrac{1+\sqrt{-\varepsilon}}{1-\sqrt{-\varepsilon}}\right)^{c/\sqrt{-\varepsilon}}, & \varepsilon < 0, \\[2ex] \dfrac{2c^2\left[\tan^{-1}\sqrt{\varepsilon} - \dfrac{\sqrt{\varepsilon}}{1+\varepsilon} - \pi\left(1 - \exp\dfrac{2\pi c}{\sqrt{\varepsilon}}\right)^{-1}\right]}{e^{2c\varepsilon^{3/2}}\left[1 - \exp\left(-\dfrac{2\pi c}{\sqrt{\varepsilon}}\right)\right]}, & \varepsilon > 0. \end{cases}$$

In the case of extremely low concentrations ($|\ln c| \gg 1$) this result proves to be especially simple (Figure 3):

$$\frac{|\kappa_0|\rho(E)}{2} \approx \begin{cases} \frac{4}{3}c^2, & \begin{cases} \varepsilon < 0,\ |\varepsilon| \ll 1, \\ 0 < \varepsilon \ll c^2/\ln^2 c, \end{cases} \\[2ex] 2\pi c^2 \varepsilon^{-3/2} \exp\left(-\dfrac{2\pi c}{\sqrt{\varepsilon}}\right), & \dfrac{c^2}{\ln^2 c} \ll \varepsilon \ll c^2, \\[2ex] \dfrac{1}{2\pi\sqrt{\varepsilon}}, & c^2 \ll \varepsilon \ll 1. \end{cases}$$

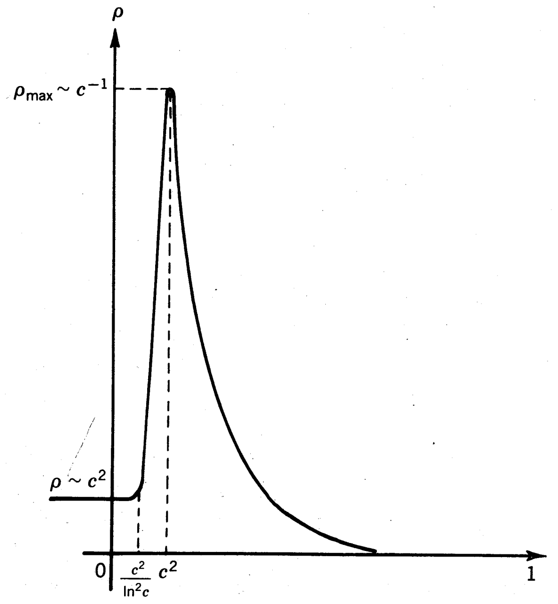

Figure 3

Thus, the main difference between the density of states in the region $|\varepsilon| \ll 1$ in the system of attractive scatterers and that in the system of repulsive scatterers lies in the presence of a flat section $\rho(E) \approx 8c^2/3|\kappa_0|$ to the left of the point $\varepsilon_0 \approx c^2/\ln^2 c$. This has a simple physical explanation. Indeed, a delta-like one-dimensional well always contains exactly one level, while two such wells that are infinitesimally close constitute a single delta-like well of a greater intensity but still containing only one level. This means that a system of paired levels generated by two delta-like scatterers merges with the initial continuous spectrum as the distance between the scatterers decreases and produces a tail in the density of states with an amplitude proportional to c^2. As we will show in Section 28, in the three-dimensional case similar arguments make it possible to observe the behavior of the density of states in a broad energy range, or practically in the entire interval from the local level to the initial continuous spectrum.

6 A METHOD FOR CALCULATING THE INTEGRAL DENSITY OF STATES BASED ON THE PHASE FORMALISM

6.1 The General Formulas for the Number of States

The starting point of the method of calculating the density of states discussed in the preceding section was Eq. (1.36), which expresses $\rho(E)$ in terms of the Green function and is valid in a space of any dimensionality and for an arbitrary interaction. The special features of the problem (the fact that it is one-dimensional and only the nearest neighbors are included in the interaction) were manifested only in calculating $\langle G_{00}(E)\rangle$. However, in the one-dimensional case there exists a considerably closer link between $\rho(E)$ and the solutions of the equation that defines the random operator considered. For a continuous model this interrelationship can be formulated as follows: The number of states with an energy not exceeding E in a system that occupies the interval $(0, L)$ can at the most be greater by unity than the number of zeros on $(0, L)$ possessed by the solution of the corresponding Schrödinger equation with a given logarithmic derivative at one of the ends. This theorem [known as the node-counting theorem; see Courant and Hilbert (1953), and Mandel'shtam (1972)] enables finding a number of relationships for the density of states in the one-dimensional case that are simpler than those discussed above. (The connection between the spectrum of the dynamical equation and the zeros of the solution of this equation was first used by Schmidt (1957) for the range of problems examined here.) It also appears that the relationships remain valid even when the node-counting theorem, in the form stated here, may become invalid, for instance, when the number of states differs from the number of zeros by a quantity that is energy-dependent and grows more slowly than L as $L \to \infty$. For this reason we will derive the abovementioned relations without resolving to the node-counting theorem. Instead we base our proof on the basic properties of the Schrödinger equation, properties that enable proving similar relationships in other situations, such as in discrete cases and for the one-dimensional Dirac equation (see Sections 7 and 8).

Let $\mathcal{N}_L(E_1, E_2)$ be the number of states of the Schrödinger equation

$$-\psi'' + U(x)\psi = E\psi, \qquad 0 < x < L, \tag{6.1}$$

with energies lying in the interval (E_1, E_2), $E_1 < E_2$. Then

$$\mathcal{N}_L(E_1, E_2) = \int_{E_1}^{E_2} \rho_L(E')\, dE'.$$

Next we introduce the phase $\alpha(x)$ of the wave function thus:

$$\cot \alpha(x) = \frac{\psi'(x)}{\psi(x)}.$$

To find $\alpha(x)$ unambiguously we must require that it be a continuous function of x [we assume that $U(x)$ has no delta-like singularities; the required modifications in case it does have such singularities are given below]. From (6.1) we find that $\alpha(x)$, which we call the nonreduced phase, satisfies the equation

$$\alpha' = \Phi_E(U, \alpha), \qquad \Phi_E(U, \alpha) = \cos^2\alpha + [E - U(x)]\sin^2\alpha, \tag{6.2}$$

and the initial condition $\alpha(0) = \alpha_0$, with $\cot \alpha_0 = (\psi'/\psi)_{x=0}$, $0 \le \alpha_0 < \pi$. If we differentiate Eq. (6.2) with respect to E and integrate the resulting linear equation, we find that

$$\frac{\partial \alpha}{\partial E} = \int_0^x \sin^2\alpha(y) \left\{ \exp\left(-\int_y^x [U(t) - E + 1] \sin 2\alpha(t)\, dt \right) \right\} dy.$$

Thus

$$\frac{\partial \alpha(x, E)}{\partial E} > 0.$$

This relationship indicates that the phase is a monotonic function of the energy; it is the main relationship in the formalism that we are developing here. Indeed, it immediately implies that the number per unit length, $\mathcal{N}_L(E_1, E_2)$, of eigenvalues of the Schrödinger equation in $(0, L)$ with the boundary conditions $\cot \alpha(0) = \cot \alpha_0$ and $\cot \alpha(L) = \cot \alpha_L$ coincides with the number of values of E that belong to the interval (E_1, E_2) and for which $\alpha(L, E) = \alpha_L + m\pi$, where m is an integer; this number is equal to

$$\mathcal{N}_L(E_1, E_2) = L^{-1}\left[\frac{\alpha(L, E_2) - \alpha(L, E_1)}{\pi} \right],$$

with $[x]$ the integer part of x. Passing to the limit as $L \to \infty$ and allowing for the fact that the number of states is a self-averaged quantity (see Section 3), we arrive at the following relation:

$$\mathcal{N}(E_1, E_2) = \lim_{L\to\infty} (\pi L)^{-1}\langle \alpha(L, E_2) - \alpha(L, E_1)\rangle. \tag{6.3}$$

Integrating Eq. (6.2) with respect to x from 0 to L and substituting the expression for $\alpha(L, E)$ into (6.3), we can rewrite this relation in the following form:

$$\mathcal{N}(E_1, E_2) = \pi^{-1}\left(\langle \Phi_{E_2}(U, \alpha)\rangle_{\mathrm{st}} - \langle \Phi_{E_1}(U, \alpha)\rangle_{\mathrm{st}}\right), \tag{6.4}$$

where on the right-hand side we have averages over the joint stationary distribution of the random potential $U(x)$ and phase $\alpha(x)$ reduced to the interval $(0, \pi)$.

Now let us take the function $P_E(x, \alpha) = \langle \delta_\pi(\alpha(x, E) - \alpha)\rangle$, where $\delta_\pi(\alpha) = \sum_{n=-\infty}^{\infty} \delta(\alpha - n\pi)$ is a π-periodic delta function. The function $P_E(x, \alpha)$ is the probability density for the distribution of the reduced phase α. By means of (6.2) we can find that

$$\begin{aligned}\frac{\partial P_E(x, \alpha)}{\partial x} &= \left\langle \delta_\pi'(\alpha(x, E) - \alpha)\frac{\partial \alpha(x, E)}{\partial x}\right\rangle\\ &= -\frac{\partial}{\partial \alpha}\langle \delta_\pi(\alpha(x, E) - \alpha)\Phi_E(U, \alpha)\rangle,\end{aligned}$$

or

$$\frac{\partial P_E(x, \alpha)}{\partial x} + \frac{\partial J_E(x, \alpha)}{\partial \alpha} = 0, \tag{6.5}$$

where

$$J_E(x, \alpha) = \langle \delta_\pi(\alpha(x, E) - \alpha)\Phi(U(x), \alpha(x, E))\rangle \tag{6.6}$$

is the probability flux, while (6.5) is simply a continuity equation, which expresses the probability conservation law, that is, the fact that the normalization of $P_E(x, \alpha)$ remains constant in time, or

$$\int_0^\pi P_E(x, \alpha)\, d\alpha = 1.$$

As $x \to \infty$, the flux $J_E(x, \alpha)$ tends to its limiting value J_E, which depends neither on x [in view of the spatial homogeneity] nor on α [in view of (6.5)]:

$$J_E = \lim_{x\to\infty} \pi^{-1}\int_0^\pi J_E(x, \alpha)\, d\alpha = \pi^{-1}\langle \Phi_E(U, \alpha)\rangle_{\mathrm{st}}.$$

Combining this expression for J_E with (6.4) enables us to express $\mathcal{N}(E_1, E_2)$ directly in terms of the "stationary" probability flux J_E:

$$\mathcal{N}(E_1, E_2) = J_{E_2} - J_{E_1}. \tag{6.7}$$

If we combine this with (6.6) and the fact that $J_E = \lim_{x\to\infty} J(x, \alpha)$ for any value of α, we find that

$$\mathcal{N}(E_1, E_2) = \langle \Phi_{E_2}(U, 0)\rangle_{\mathrm{st}} - \langle \Phi_{E_1}(U, 0)\rangle_{\mathrm{st}}. \tag{6.8}$$

Here in order to calculate the averages we must know the stationary distributions of U and α only at $\alpha = 0$.

We have assumed that the potential has no delta-like singularities. But, as one can easily see by passing to the limit of narrow-peaked but smooth potentials, the above formulas (6.3) and (6.4) for $\mathcal{N}(E_1, E_2)$ remain valid when $U(x)$ does have such singularities [e.g., for potentials of the type (1.7) with $u(x) = \kappa_0\delta(x)$]. However, the reader must bear in mind that now $\alpha(x)$ undergoes jumps where the delta functions are concentrated. The magnitudes of such jumps are determined by Eq. (6.2) and the requirement that the values of the phase to the left and to the right of such points lie on one branch of the cotangent [in one interval $(m\pi, (m+1)\pi)$].

Thus, the formulas (6.3) and (6.4) are universal and valid for any random potential. They show that to calculate $\mathcal{N}(E_1, E_2)$ and hence $\rho(E)$ we must know the joint probability distribution of the potential and reduced phase or the probability distribution of the nonreduced phase when $x \to \infty$. These functions in many cases satisfy certain integral equations (of the Smoluchowski type) or differential equations (of the Fokker–Planck type), equations that describe the probabilistic evolution of $U(x)$ and $\alpha(x, E)$ and can be derived on the basis of (6.2) with allowance for the probabilistic properties of the random potential. Therefore the form of these equations is not universal [e.g., see Eqs. (6.14), (6.33), (6.55), (6.69) and (6.90)], so that the methods by which they are derived and investigated are specific for particular cases or for groups of similar cases. These questions are discussed in sections of this chapter that follow.

The universal formulas (6.3) and (6.4) may sometimes be awkward in concrete calculations, since the first contains explicitly the operation of taking the limit and the second requires calculating the integral $\int \Phi(U, \alpha)P_E(U, \alpha)\, dU\, d\alpha$ in addition to knowing the stationary probability density $P_E(U, \alpha)$ of $U(x)$ and $\alpha(x, E)$ when calculating $\mathcal{N}(E_1, E_2)$. However, if one has additional information about the random potential, simpler expressions for $\mathcal{N}(E_1, E_2)$ can be obtained. For instance, if we know that $U(x)$ has no delta-like singularities, then using (6.8) and (6.2) (in the latter $\alpha = 0$) instead of (6.4) yields

$$\mathcal{N}(E_1, E_2) = P_{E_2}(0) - P_{E_1}(0), \tag{6.9}$$

where $P_E(\alpha)$ is the stationary probability density of the reduced phase only.

This result is remarkable because it enables finding the number of states directly from $P_E(\alpha)$, that is, without any additional operations.

In many interesting cases (see Sections 6.2, 6.4, 6.8, and 8) we will also use (6.7) to find the number of states. This formula is especially convenient when one is able to find an explicit expression for the probability density flux $J_E(x, \alpha)$ or when the stationary flux $J(E)$ enters explicitly into the equations of the problem. If, in addition, instead of the reduced phase α we use another variable $f(\alpha)$, substitution of this variable into Eq. (6.5) shows that the $J(E)$ in (6.7) is defined thus:

$$J(E) = \lim_{x\to\infty} J_E(x, f(\alpha))\operatorname{sign} f'(\alpha), \tag{6.10}$$

where $J_E(x, f(\alpha))$ is the probability density flux of $f(\alpha)$.

6.2 The White-Noise Potential

We start from the simplest case, where $U(x)$ in (6.1) represents white noise (Halperin, 1965), that is, is given by a delta-correlated random function with a zero mean:

$$\overline{U} = 0, \qquad \langle U(x)U(x')\rangle = 2D\delta(x - x').$$

The generating functional of this potential has the form (2.6) with

$$B(x) = 2D\delta(x), \tag{6.11}$$

while the potential, in accordance with the results of Section 2.2, possesses the properties of spatial homogeneity and the disappearance of correlations at infinity.

Let us introduce a new dynamical variable

$$z = \cot\alpha = \frac{\psi'}{\psi}, \tag{6.12}$$

which according to (6.1) satisfies the equation

$$z' = -(z^2 + E) + U(x). \tag{6.13}$$

For the probability density $P(x, z)$ of z at point x we can write the Fokker–Planck equation [the derivation of this equation for a situation that is more general than (6.11) will be described in Section 10; see also Klyatskin (1975)]

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial z}\left((z^2 + E)P + D\frac{\partial P}{\partial z}\right), \tag{6.14}$$

which is a particular case of the continuity equation (6.5) with a flux density

$$J_E(x, z) = -(z^2 + E) - D\frac{\partial P}{\partial z}.$$

Sending x to infinity and bearing in mind that $z'(\alpha)$ is negative, we arrive [in accordance with (6.10)] at the following formula:

$$J(E) = (z^2 + E)P + D\frac{dP}{dz}, \tag{6.15}$$

where $P(z)$ is the stationary probability density of z and satisfies Eq. (6.14) with a zero left-hand side. The solution of this equation can readily be found:

$$\frac{DP(z)}{J(E)} = \exp\left(-\frac{z^3}{3D} - \frac{Ez}{D}\right)\int_{-\infty}^{z}\exp\left(\frac{t^3}{3D} + \frac{Et}{D}\right)dt, \tag{6.16}$$

with $J(E)$ found from the normalization of $P(z)$ to unity, or

$$J^{-1}(E) = D^{-1/3}\int_{-\infty}^{\infty}dx\int_{-\infty}^{x}dy\exp[\Phi(y) - \Phi(x)], \tag{6.17}$$

$$\Phi(x) = \frac{x^3}{3} + \frac{Ex}{D^{2/3}}.$$

The first to obtain this result were Frish and Lloyd, 1960, who proceeded from the potential (5.16) to white noise by way of the limiting process $\kappa_0 \to 0$, $\bar{r} \to 0$, $\kappa_0^2\bar{r}^{-1} \sim 1$. Halperin, 1965, discussed the calculation of not only the number of states but also the spectral density (1.30) and conductivity (1.32) for the white-noise potential.

Equation (6.17) can be rewritten as

$$J^{-1}(E) = \pi^{1/2}D^{-1/3}\int_0^{\infty}x^{-1/2}\exp\left(-\frac{x^3}{12} - \frac{Ex}{D^{2/3}}\right)dx, \tag{6.18}$$

from which we find that

$$J(E) \approx \pi^{-1}|E|^{1/2}\exp\left(-\frac{4}{3}\frac{|E|^{3/2}}{D}\right)\left[1 + O\left(\frac{D}{|E|^{3/2}}\right)\right],$$

$$E < 0, \quad |E| \gg D^{2/3}, \tag{6.19}$$

$$J(E) \approx \pi^{-1}E^{1/2}\left(1 + \frac{5}{32}\frac{D^2}{E^3} + O\left(\frac{D^4}{E^6}\right)\right),$$

$$E > 0, \quad E \gg D^{2/3}. \tag{6.20}$$

According to (6.17), $J(E)$ tends to zero as $E \to -\infty$, and therefore $J(E) = \mathcal{N}(E)$, where $\mathcal{N}(E) = \mathcal{N}(E, -\infty)$ is the number of states with energies from $-\infty$ to E per unit length in an infinite system.

Note that if in Eq. (6.15) we go over to Fourier transforms, we can arrive at an expression for the number of states $\mathcal{N}(E)$ that differs from (6.18) (Halperin, 1965):

$$\mathcal{N}(E) = D^{1/3}\pi^{-1}\left[\mathrm{Ai}^2\left(-\frac{E}{D^{2/3}}\right) + \mathrm{Bi}^2\left(-\frac{E}{D^{2/3}}\right)\right]^{-1}, \qquad (6.21)$$

where $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ are the linearly independent solutions of the Airy differential equation

$$y'' - xy = 0.$$

6.3 Spectrum Properties in the Vicinity of the Mean Value of the Potential

If the peaks in the random potential are high and the correlation radius small, then at an energy close to the mean value of the potential the latter will closely resemble white noise. For this reason the model discussed in Section 6.2 may provide, in accordance with the general reasoning concerning approximate models discussed in Section 1.1, a fairly good description of the spectrum properties in the vicinity of the mean value of the potential. In this case the number of states

$$\mathcal{N}(E_1, E_2) = J(E_2) - J(E_1)$$

with energies in the vicinity of the mean value of the potential can be obtained from (6.18). Here we present arguments that will enable us to see what properties the potential must have (the magnitude of fluctuations and the correlation radius) so that it can be replaced by white noise, and in what energy range such a replacement is possible. Moreover, later we will see that for potentials satisfying the conditions that we will now formulate, the white-noise model correctly describes, in the vicinity of the mean value of the potential, not only the behavior of the density of states, but also the rate of decay for the wave function's envelope, the conductivity, the localization of the wave function (Chapter 3), and the transmission coefficient of a plane monochromatic wave (Chapter 7).

As shown earlier, the density of states is determined completely by the probability distribution of the phase α of the wave function or any function of α, say z. Therefore we must only establish in what conditions the probabilistic properties of $z(x)$ determined from Eq. (6.13) with an arbitrary $U(x)$ are close to those of $z(x)$ corresponding to white noise, that is, to a delta-correlated Gaussian random function. Since the latter has a distribution symmetric with respect to the mean value, it is natural to isolate this value from the start:

$$U(x) = \bar{U} + v(x), \qquad \bar{U} = \langle U(x) \rangle, \qquad \langle v(x) \rangle = 0.$$

Now we must assume that $v(x)$ varies in the parameter range considered here much faster than the wave function $\psi(x)$, since we are looking for a delta-correlated random function. But this is possible only at low enough energies, that is, when the characteristic distances over which $\psi(x)$ varies (the de Broglie wavelength and the bound-state radius) are much larger than the potential's correlation radius. Since these distances decrease as the absolute value of the energy grows, it is clear that a delta-correlated potential is a good approximation of the initial potential when the energy is close to the mean value of the potential. If we denote the correlation radius of $v(x)$ by r_c, which is the distance over which the correlation function

$$B_v(x) = \langle v(x) v(0) \rangle \qquad (6.22)$$

decreases appreciably, then the corresponding energy region can be determined by the following inequality:

$$|E - \bar{U}| \ll r_c^{-2}. \qquad (6.23)$$

To establish the parameter that determines the order of magnitude of the potential fluctuations that provide the main contribution to the spectrum in this energy region, we note that

$$2D = \int_{-\infty}^{\infty} B_v(x)\, dx \qquad (6.24)$$

remains finite when we go over to white noise. Since a Gaussian random function is determined entirely by its correlation function, which in the case of white noise has the form $B(x) = 2D\delta(x)$ [see (6.11)] and contains only one parameter D, this parameter is the only dimensional parameter of the problem. This means that the fluctuations are of the order of $D^{2/3}$ and hence it is natural to change variables in Eq. (6.13)* in the following manner:

$$D^{1/3}x = t, \qquad D^{1/3}z = \zeta.$$

As a result we arrive at the equation

$$\dot{\zeta} = -\zeta^2 - (E - \bar{U})D^{-2/3} + D^{-2/3}v(tD^{-1/3}). \qquad (6.25)$$

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*In nonlinear mechanics the variable t is often called the slow time; see Mitropol'skiĭ (1971) and Stratonovich (1961).