

Conservation laws.

Reading: Schwinger, Chap. 3, 4 (or Jackson, Chap. 5, 6)

1. Poynting vector in a conductor.

Consider a long cylindrical wire of radius a carrying a steady current I . The conductivity of the material is σ . The magnetic field of the current is transverse to the electric field, which gives rise to the energy flux. Find the Poynting vector $\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{B}$ outside and inside the wire, at $r > a$ and $r < a$. Calculate the flux of \mathbf{S} through a cylinder of radius r coaxial with the wire, compare with the Joule heat produced by current, and verify the energy conservation in this system.

2. Poynting vector in a transmission line.

A circular cylindrical transmission line is made of two coaxial cylinders with the radii a and b , $a < b$. The outer cylinder carries current I which returns along the inner cylinder; the cylinders' charge per unit length is λ and $-\lambda$, respectively. Find the electric and magnetic field, and determine the energy flux, given by the Poynting vector \mathbf{S} . By integrating \mathbf{S} over the cross-section, obtain the total energy flux in the transmission line.

By a calculation show that, if a resistor R is connected to an end of the line, the power dissipated equals the total energy flow due to \mathbf{S} .

3. Maxwell electric stress tensor

Forces in electrostatics can be obtained using the stress tensor of the electric field

$$T_{ij} = \frac{1}{8\pi} \delta_{ij} E^2 - \frac{1}{4\pi} E_i E_j.$$

As the following two examples illustrate, this approach becomes especially convenient for problems with complicated geometry and/or distributed charge density.

a) Show that the pressure of electric field on a surface of a charged conductor is equal to $\mathbf{E}^2/8\pi$, where \mathbf{E} is the field near the surface. Obtain the same answer by using the formula $\mathbf{F} = \sum_i q_i \mathbf{E}(\mathbf{r}_i)$ (be careful with the factors of two).

b) Consider a spherical ball of charge of radius R carrying total charge Q . The ball is cut into two halves by a horizontal plane along the equator. By using the stress tensor, find the force that the lower half exerts on the upper half.

4. Maxwell magnetic stress tensor

A long conducting cylinder of radius a carries a DC current I , which is uniformly distributed over the surface and is flowing along the cylinder axis.

a) Using the stress tensor of the magnetic field,

$$T_{ij} = \frac{1}{8\pi} \delta_{ij} B^2 - \frac{1}{4\pi} B_i B_j$$

find the pressure of the field on the cylinder surface.

b) Calculate the pressure by using the Lorentz force on a wire, $d\mathbf{F} = \frac{1}{c} I d\mathbf{l} \times \mathbf{B}$. (Again, be careful with the factors of two)

5. One-dimensional waves.

Consider the wave equation in one dimension,

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} = 0 \quad (1)$$

This equation describes waves in a transmission line, such as that of Problem 2, with the identification $\partial f/\partial t \equiv E$, $\partial f/\partial x \equiv B$, where the fields E , B correspond to the so-called TEM mode and $v = c$.

- Show that the general solution of Eq.(1) is of the form $f(x, t) = f_R(x - vt) + f_L(x + vt)$ describing wavepackets moving to the right and to the left with velocity v . (Hint: use factorization $\partial^2/\partial x^2 - \frac{1}{v^2}\partial^2/\partial t^2 = (\partial/\partial x - \frac{1}{v}\partial/\partial t)(\partial/\partial x + \frac{1}{v}\partial/\partial t) = 4\partial^2/\partial x_+\partial x_-$, where $x_{\pm} = x \pm vt$)
- Given the initial conditions $\partial f/\partial x|_{t=0} = F_1(x)$ and $\partial f/\partial t|_{t=0} = F_2(x)$, determine the right- and left-propagating components f_L and f_R at $t > 0$.
- Consider the energy density defined by analogy with $U = \frac{1}{8\pi}(E^2 + B^2)$ as

$$U = \frac{1}{2v^2} \left(\frac{\partial f}{\partial t} \right)^2 + \frac{1}{2} \left(\frac{\partial f}{\partial x} \right)^2$$

From Eq.(1) derive energy conservation law $\partial U/\partial t + \partial S/\partial x = 0$ with the function $S = -(\partial f/\partial t)(\partial f/\partial x)$ identified with energy flux.

- Derive the equivalent of momentum conservation, $\partial S/\partial t + \partial T/\partial x = 0$. What is the ‘stress tensor’ function T in this case?

6. Electromagnetic waves.

Consider solutions of Maxwell’s equations in the absence of charges and currents, and such that the fields \mathbf{E} and \mathbf{B} depend on one spatial variable z , and on time t .

- Show that the equations decouple into two pairs of equations for the components E_y and B_x , and for E_x and B_y .
- In each case, by eliminating one of the fields (say E), derive a wave equation.
- Find the relation between the amplitudes E and B which corresponds to waves propagating in a specific direction, $+\hat{\mathbf{z}}$ or $-\hat{\mathbf{z}}$.

7. Electromagnetic pulses.

The electric and magnetic fields at $t = 0$ are

$$\mathbf{E} = E_0 f(z) \hat{\mathbf{x}}, \quad \mathbf{B} = B_0 f(z) \hat{\mathbf{y}}, \quad f(z) \propto \frac{1}{z^2 + a^2}. \quad (2)$$

- Find the electromagnetic energy and momentum density and their ratio.
- Show that at $t > 0$ there are two wavepackets travelling in the directions $\hat{\mathbf{z}}$ and $-\hat{\mathbf{z}}$.
- Determine the fraction of the initial energy of the field which is carried in the $\hat{\mathbf{z}}$ and $-\hat{\mathbf{z}}$ direction. At what ratio $\alpha = E_0/B_0$ there is (i) just one wavepacket; (ii) two wavepackets of equal intensity?