8.334: Statistical Mechanics II Problem Set # 6

Fluctuations in low space dimension.

1. 1D Ising model, correlation length, susceptibility

a) Consider a pair spin correlation function in the 1D Ising model $\mathcal{H} = \sum_i K s_i s_{i+1}$, $s_i = \pm 1$. *This problem can be solved by the transfer matrix method, discussed in recitation, or by the method of auxiliary link variables* $\xi_i = s_i s_{i+1}$, *discussed in Lecture 6 [Eqs.(10,11,12)]. Show that in terms of the link variables, the correlation function is*

$$\langle s_{i_1} s_{i_2} \rangle = \langle \prod_{i_1 \le j < i_2} \xi_j \rangle , \quad i_1 < i_2. \tag{1}$$

Using the hamiltonian for ξ*'s (Eq.(11), Lec. 6), average the product (1). Comparing to the asymptotic formula* $C_2(x - x') \propto \exp(-|x - x'|/\xi)$, *determine the correlation length* ξ *as a function of temperature.*

In terms of the link variables the Hamiltonian is

$$H = K \sum_i s_i s_{i+1} = K \sum_i \xi_i. \tag{2}$$

Hence

$$\langle s_i s_{i'} \rangle = \langle \prod_{i \le j < i'} \xi_j \rangle = \left(\frac{e^K - e^{-K}}{e^K + e^{-K}} \right)^{|i-i'|} = (\tanh K)^{|i-i'|} = e^{-|i-i'|/\xi} \tag{3}$$

Therefore

$$\xi = -1/\ln \tanh K = \begin{cases} e^{2J/T}, \, T \ll J \\ \frac{1}{\ln(T/J)}, \, T \gg J \end{cases} \tag{4}$$

b) At distances much larger than ξ *the system is disordered, while at distances less than* ξ *the spins are strongly correlated. In RG language, for block sizes* $\geq \xi$ *the block spins are effectively decoupled from each other. (Indeed, renormalized temperature becomes very large at the length scale* ξ, *indicating decoupling.)*
Consider spin susceptibility $\chi = dm/dh|_{h=0}$. *Treating the problem as a system of independent block spins of size* ξ, *each taking values* $\pm\xi$, *show that* $\chi \sim \xi/T$.

For one block of size ξ the spin susceptibility is $\chi_{block} \sim \xi^2/T$ (Curie's law for spin $s = \pm\xi$. Multiplying this by the block density ξ^{-1}, obtain $\chi \propto \xi/T$.

c) It is interesting to look at the susceptibility in the ξ *variables representation. Start with the general expression* $\chi = d^2 \ln Z/dh^2|_{h=0}$, *and write the susceptibility in terms of* ξ_i. *Average over* ξ*'s using the method of part a) and compare the result with the above estimate for independent block spins.*

$$\chi = \frac{dm}{dh} = \frac{d^2 \ln Z}{dh^2} = \frac{1}{T} \sum_{i,i'} \langle s_i s_{i'} \rangle = \frac{1}{T} \sum_{ii'} (\tanh K)^{|i-i'|} = \frac{L}{T} \left(\frac{2}{1 - \tanh K} - 1 \right) = \frac{e^{2K}}{T} L \tag{5}$$

where L is the system size. This agrees with the result of part b).

2. Magnetization and susceptibility of a 2D Heisenberg ferromagnet

a) Use the thermodynamic potential of spin excitations of a 2D Heisenberg ferromagnet at low temperature (Problem 2, PS#1), and find magnetization $m = d\ln Z/dh$ as a function of magnetic field h. Show that the zero temperature value $m(0)$ is reduced by thermal fluctuations so that the difference $\delta m = m(0) - m(T)$ diverges logarithmically as $h \to 0$.

The free energy is

$$F(h) = T\int \ln\left(1 - e^{-\beta\omega(k)}\right)\frac{d^2k}{(2\pi)^2} - |m_0|h \tag{6}$$

where $\omega(k) = h + Jk^2$ is the spin wave spectrum. The first term in the expression for the free energy is the contribution of thermal fluctuation and the second term with $|m_0| = 1$ is the is the ground state energy.

The average magnetization is

$$m = -\frac{dF}{dh} = 1 - T\int \frac{\beta e^{-\beta\omega(k)}}{1 - e^{-\beta\omega(k)}}\frac{d\omega(k)}{dh}\frac{d^2k}{(2\pi)^2} = 1 - \int \frac{1}{e^{\beta\omega(k)} - 1}\frac{d^2k}{(2\pi)^2} \tag{7}$$

Consider the contribution of the small k excitations, $\omega(k) \le T$:

$$m = 1 - \frac{T}{4\pi^2}\int_0^{\omega(k)<T}\frac{d^2k}{\omega(k)} = 1 - \frac{T}{2\pi}\int_0^{\sqrt{T/J}}\frac{k\,dk}{h + Jk^2} = 1 - \frac{T}{4\pi J}\ln\frac{T}{h} \tag{8}$$

where we assumed that h is small, $h \ll T$. Therefore the difference $\delta m = m(0) - m(T)$ diverges logarithmically, $\delta m \propto \ln(T/h)$

b) Relate the divergence found in part a) with the logarithmic divergence from the Mermin and Wagner theorem. The latter is concerned with transverse magnetization fluctuations, so one has to connect the depletion of spin polarization along the field with its transverse fluctuations.

Introduce transverse magnetization $\delta m_\perp$. Then

$$m_z = \sqrt{1 - \delta m_\perp^2} = 1 - \frac{1}{2}\delta m_\perp^2 + O(\delta m_\perp^4) \tag{9}$$

Using Mermin-Wagner theorem (see comment in Lecture 8) find

$$\langle \delta m_\perp^2\rangle = \frac{2T}{(2\pi)^2}\int \frac{d^2k}{Jk^2 + h} = \frac{T}{2\pi J}\ln\frac{T}{h} \tag{10}$$

Hence obtain for m_z the same expression as in part a).

c) It is interesting to consider the zero field susceptibility of this system. From RG analysis of the nonlinear sigma model, we know that the effective coupling becomes very small at some temperature-dependent length scale $\xi(T)$. Use the arguments similar to those in part b) of Problem 1 to estimate the susceptibility. Does your estimate agree with the above result for $m(T)$ extrapolated to $h = 0$?

For blocks of size $\xi = a\exp(2\pi/g)$, the effective spin is $s \sim \xi^2$. Thus the susceptibility is

$$\chi \approx \frac{1}{T}(\xi^2)^2\xi^{-2} = \frac{\xi^2}{T} = \frac{a^2}{T}e^{4\pi/g} \tag{11}$$

since $a^2 \approx J/T$, the susceptibility is

$$\chi \approx \frac{J}{T^2} e^{4\pi J/T}. \tag{12}$$

Compare this to the results of parts a) and b)

$$\chi(h) = \frac{T}{4\pi J}\frac{1}{h} \tag{13}$$

This expression is valid at sufficiently large h, so that the logarithmic term $\ln(T/h)$ is small: $T/h < \exp(4\pi J/T)$. Hence $h > T\exp(-4\pi J/T)$. Using this estimate in the expression for $\chi(h)$, obtain

$$\chi(0) \approx \frac{1}{4\pi J} e^{4\pi J/T} \tag{14}$$

The large exponential factor is reproduced correctly, but the prefactor is different by $(T/J)^2$.

3. The sine-Gordon model

Consider a 2D gaussian problem of a scalar field $\phi(\mathbf{x})$ perturbed by a small cos *term:*

$$\mathcal{H} = \iint \left(\frac{1}{2}(\partial_\mu \phi)^2 + \lambda \cos(\beta\phi)\right) d^2x \tag{15}$$

Here λ is a coupling constant, and β is a dimensionless parameter.
Apply the field-theoretic RG transformation described in Lecture 8 (coarse grain + rescale + renormalize hamiltonian) to this problem, treating the cos *term as a perturbation. On each RG step, split the field ϕ into the fast and slow component, and average the Hamiltonian*[1] *(15) over the fast fluctuations, assuming them to be described by the quadratic part of (15).*
a) Show that the hamiltonian form is preserved upon RG and derive the RG flow equation, first order in a weak coupling constant:

$$d\lambda/dt = -f(\beta)\,\lambda\,, \quad t \equiv \ln(\Lambda/\Lambda') \tag{16}$$

Find the function $f(\beta)$ and show that, depending on the value of β, the coupling λ either decreases, or increases. For λ decreasing, the problem turns into a freely fluctuating gaussian field, corresponding to a disordered state, whereas for λ increasing, the fluctuations freeze at a certain length scale (estimate it), corresponding to the ordered state.

RG flow can be obtained by averaging $\lambda\cos\beta\phi$ over fluctuations with $\Lambda' < q < \Lambda$:

$$\langle \lambda\cos\beta\phi\rangle = \frac{1}{2}\lambda\langle e^{i\beta\phi_{slow}+i\beta\delta\phi} + c.c.\rangle = \frac{1}{2}\lambda\left(e^{i\beta\phi_{slow}} e^{-\frac{1}{2}\beta^2\langle\delta\phi^2\rangle} + c.c.\right) \tag{17}$$

where

$$\langle\delta\phi^2\rangle = \sum_q \frac{1}{q^2} = \frac{1}{2\pi}\ln\Lambda/\Lambda'. \tag{18}$$

[1]It would be more appropriate to average the exponent $e^{-\mathcal{H}}$, which is a slightly more difficult task. However, to the first order in λ the result is the same, so averaging the Hamiltonian in this problem is sufficient. The reason for this, as we discuss later, is that "the first loop RG approximation" is nonzero, and so the transition arises already in the lowest order of perturbation theory. Wait till PS#7 to see an example of RG based on averaging $e^{-\mathcal{H}}$.

Hence obtain

$$\langle \lambda \cos \beta\phi \rangle = \left(\frac{\Lambda'}{\Lambda}\right)^{\beta^2/4\pi} \lambda \cos \beta\phi_{slow}. \tag{19}$$

After rescaling $x \to \Lambda/\Lambda' x$, obtain

$$\lambda' = \left(\frac{\Lambda}{\Lambda'}\right)^{2-\frac{\beta^2}{4\pi}} \lambda. \tag{20}$$

Near fixed point $\beta^2 = 8\pi$, variation of λ is slow and one can cast the RG in a differential form:

$$\frac{d\Lambda}{dl} = \left(2 - \frac{\beta^2}{4\pi}\right) \lambda. \tag{21}$$

Solution to this ODE reproduces the result (20).

b) Add a perturbation $\mu \cos(\beta'\phi)$ to the problem (15). Show that this perturbation is irrelevant near the phase transition found in part a) when $\beta' > \beta$.
On the contrary, a perturbation with $\beta' < \beta$ is relevant. How does it affect the phase transition?

The perturbation $\mu \cos \beta'\phi$ is renormalized as

$$\frac{d\mu}{dl} = \left(2 - \frac{(\beta')^2}{4\pi}\mu\right), \text{ or } \mu' = \left(\frac{\Lambda}{\Lambda'}\right)^{2-\frac{(\beta')^2}{4\pi}} \mu. \tag{22}$$

If $\beta' > \beta$, the perturbation decreases near the critical point $\beta^2 = 8\pi$. Call it irrelevant. If $\beta' < \beta$, μ grows much faster than λ at β bellow $\sqrt{8\pi}$. Critical point $\beta^2 = 8\pi$ is destroyed and replaced by another critical point $(\beta')^2 = 8\pi$, controlled by $\mu \cos \beta'\phi$ term.

4. Magnetization in the XY problem.
Consider an XY model for planar ferromagnet in a weak in-plane magnetic field. The magnetization is described by a 2-component unit vector $\mathbf{n} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}$.
a) Show that the hamiltonian, written in terms of the angular variable θ, has the form

$$\mathcal{H} = \iint \left(\frac{1}{2} J(\partial_\mu \theta)^2 - h \cos\theta\right) d^2x \tag{23}$$

(here the field h is applied along the x axis). The hamiltonian is formally identical to the sine-Gordon problem (15), except that in the XY case the variable θ lives on a circle, not on a line. For the moment, however, let us ignore this difference.

Starting from

$$H = \int \left(\frac{1}{2} J(\partial_\mu n)^2 - \mathbf{h} \cdot \mathbf{n}\right) d^2x \tag{24}$$

and introducing phase θ: $n_1 = \cos\theta$, $n_2 = \sin\theta$, obtain

$$H = \int \left(\frac{1}{2} J(\partial_\mu \theta)^2 - h \cos\theta\right) d^2x \tag{25}$$

where $h \parallel x$.

b) The magnetization induced by the field h is given by $\bar{m} = \langle \cos\theta \rangle$. Consider the limit of a weak field, when thermal fluctuations suppress magnetization. Assume that the temperature is below the Kosterlitz-Thouless temperature, so that vortices can be excluded from the analysis.
To find the magnetization, go back to the RG analysis of Problem 3 a). At low temperature, when $f(\beta) < 0$ and coupling grows under renormalization, determine the length scale Λ_* at which the second term of the hamiltonian (23) reaches the first term. Argue that at larger length scales the fluctuations freeze out and the system becomes ordered. By averaging $\cos\theta$ over the fluctuations with wavelengths up to Λ_*, find the magnetization as a function of magnetic field.

Using the results of Problem 3, find

$$m_{\Lambda'} = \langle \cos\theta \rangle = \left(\frac{\Lambda'}{\Lambda}\right)^{1/4\pi K} \tag{26}$$

where $K = J/T$. Look for scale Λ' at which $h\cos\theta$ becomes of the order of the first term. At this scale (see Problem 3)

$$h_{\Lambda'} = \left(\frac{\Lambda}{\Lambda'}\right)^{2-\frac{1}{4\pi K}} h_{\Lambda}. \tag{27}$$

Therefore the condition on Λ is

$$h_{\Lambda'} = \left(\frac{\Lambda_0}{\Lambda'}\right)^{2-\frac{1}{4\pi K}} h \approx K. \tag{28}$$

Hence,

$$\Lambda_* = \Lambda_0 (h/K)^{\alpha}, \text{ where } \alpha = \frac{1}{2 - \frac{1}{4\pi K}}. \tag{29}$$

The fluctuations become massive at length scales bigger then $(\Lambda')^{-1}$ and freeze out. Therefore the magnetization can be estimated as

$$m_{\Lambda_*} \approx (h/K)^{\alpha/4\pi K} = (h/K)^{\gamma}, \text{ where } \gamma = \frac{1}{8\pi K - 1} \tag{30}$$