8.334: Statistical Mechanics II   Problem Set # 4

Real space renormalization. Scaling theory.

1. Renormalization for 1D Ising model

Consider a 1D Ising model with the nearest neighbor interaction in external magnetic field

$$\mathcal{H} = -\sum_i (J s_i s_{i+1} + h s_i) \tag{1}$$

To apply the renormalization group to this problem, generalize the decimation procedure discussed in class to include the coupling to magnetic field. Obtain a renormalization mapping

$$J' = f_1(J,h)\,, \quad h' = f_2(J,h) \tag{2}$$

Linearize the mapping in h in order to study the renormalization flow at small h.
Draw schematically the flow in the (J,h) plane. What are the fixed points of this flow? Which of them are stable or unstable? Which states correspond to the fixed points?
Summation over s_i with $i = 2n+1$ yields

$$\sum_{s_{i+1}=\pm 1} \exp(J s_i s_{i+1} + J s_{i+1} s_{i+2} + h s_{i+1}) = \begin{cases} e^{2J+h} + e^{-2J-h} & s_i + s_{i+2} = 2 \\ e^{h} + e^{-h} & s_i + s_{i+2} = 0 \\ e^{-2J+h} + e^{2J-h} & s_i + s_{i+2} = -2 \end{cases} \tag{3}$$

where notations $\beta J \to J$ and $\beta h \to h$ are used. Compare this to

$$Z_0 \exp(J' s_i s_{i+2} + \delta h (s_i + s_{i+2})) = \begin{cases} Z_0 e^{J'+2\delta h} & s_i + s_{i+2} = 2 \\ Z_0 e^{-J'} & s_i + s_{i+2} = 0 \\ Z_0 e^{J'-2\delta h} & s_i + s_{i+2} = -2 \end{cases} \tag{4}$$

Therefore

$$e^{4\delta h} = \frac{\cosh(2J+h)}{\cosh(2J-h)} = \frac{1+\tanh 2J \tanh h}{1-\tanh 2J \tanh h} \tag{5}$$

$$e^{4J'} = \frac{\cosh(2J+h)\cosh(2J-h)}{\cosh^2 h} = \cosh^2(2J) - \sinh^2(2J)\tanh^2 h \tag{6}$$

To the order $O(h)$, obtain

$$2\delta h = \tanh(2J) h \tag{7}$$

$$4J' = 2 \ln\cosh(2J) \tag{8}$$

Hence

$$h' = \left(1 + \frac{1}{2}\tanh(2J)\right) h \tag{9}$$

$$J' = \frac{1}{2}\ln\cosh(2J) \tag{10}$$

Under RG, J decreases, while h grows. Fixed points are $h = 0$, $J = 0, \infty$. There is no phase transition.

2. Migdal-Kadanoff renormalization for Ising model
a) Apply real space renormalization to Ising ferromagnet in magnetic field. Use the bond-moving algorithm, as discussed in class. Sketch the RG flow on the (J, h) plane, identify fixed points, stable and unstable, and discuss their physical meaning.
b) Calculate the temperature and magnetic field eigenvalues y_t, y_h of the RG flow at the fixed point found in part a). Assuming scaling of the singular part of the free energy,

$$f(\tau, h) = b^{-d} f(b^{y_t}\tau, b^{y_h} h) \tag{11}$$

with small h, τ, and $b = 2$ the rescaling factor, predict the form of singularity in the specific heat and susceptibility. Compare to the exact results (Eq.2, Lecture 5).
After the first stage,

$$\tilde{J}_x = 2J_x \tag{12}$$

$$\tilde{J}_y = \frac{1}{2}\ln\cosh 2J_y \tag{13}$$

$$\tilde{h} = \left(1 + \frac{1}{2}\tanh(2J_y)\right) h \tag{14}$$

and after the second,

$$J_x' = \frac{1}{2}\ln\cosh 2\tilde{J}_x \tag{15}$$

$$J_y = 2\tilde{J}_y \tag{16}$$

$$h = \left(1 + \frac{1}{2}\tanh(2\tilde{J}_x)\right) \tilde{h} \tag{17}$$

Therefore after the two steps

$$J_x' = \frac{1}{2}\ln\cosh 4J_x \tag{18}$$

$$J_y = \ln\cosh 2J_y \tag{19}$$

$$h = \left(1 + \frac{1}{2}\tanh(4J_x)\right)\left(1 + \frac{1}{2}\tanh(2J_y)\right) h \tag{20}$$

Trivial fixed points are $J = 0, \infty$. Nontrivial (and unstable) fixed point is

$$J_x^* = \frac{1}{2}\ln\cosh 4J_x^* \qquad\qquad J_y^* = \ln\cosh 2J_y^* \tag{21}$$

Hence $J_x^* = J_y^*/2$. Eigenvalues are given by

$$2^{y_t} = \frac{dJ_y'}{dJ_y} = 2\tanh 2J_y^* \qquad\qquad 2^{y_h} = \left(1 + \frac{1}{2}\tanh 2J_y^*\right)^2 \tag{22}$$

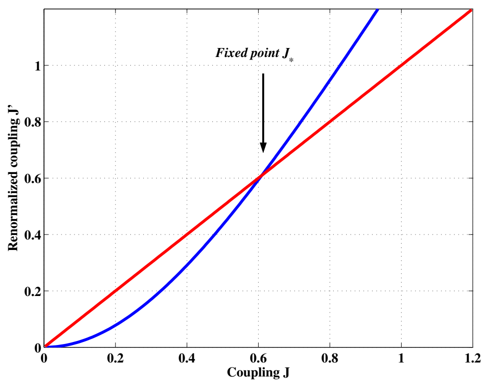

Figure 1: Renormalized coupling J' versus J for 1D Ising model.

Numerical solution (see Fig. 1) gives $J_y^* = 0.6094$ and $y_t = 0.7472$, $y_h = 1.0111$. Using results obtained in Lecture 7 with $d = 2$, obtain the critical exponents

$$\alpha = 2 - d/y_t = 2 - 2/y_t = -0.6767 \qquad \gamma = (2y_h - d)/y_t = (2y_h - 2)/y_t = 0.0297 \tag{23}$$

3. Exact renormalization on a hierarchical lattice

Renormalization group theory of a phase transition aren't necessarily approximate. Consider an Ising model on a hierarchical lattice shown in Fig. 2 and apply to it the decimation RG, as in problem 1. Find the RG transformation, identify its fixed points, and sketch the flow on the (J, h) plane.

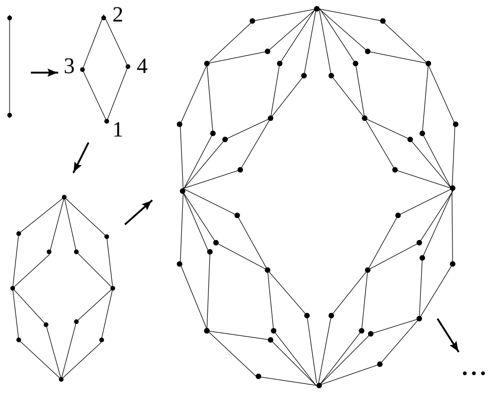

Figure 2: Hierarchical Berker lattice for which decimation can be carried out exactly.

First consider the problem at $h = 0$. Decimation is achieved by summing over the spins with two neighbors:

$$Z_\diamond = \sum_{s_1,s_2} \left(\sum_{s_3} e^{Js_3(s_1+s_2)} \right)^2 = \sum_{s_1,s_2} \left(2\cosh J(s_1+s_2)\right)^2 = 4\cosh 2J \sum_{s_1,s_2} e^{J's_1s_2} \tag{24}$$

where $\exp(J') = \cosh 2J$. The partition function for n-level lattice is

$$Z = \left(\ldots \left(\left((4\cosh 2J)^4 4\cosh 2J' \right)^4 4\cosh 2J'' \right)^4 \ldots \right)^4 4\cosh 2J^{(n)} \tag{25}$$

The recursion relations for J and $F = -T \ln Z$ are

$$J_n = \ln \cosh 2J_{n-1} \qquad\qquad F_n = 4F_{n-1} - T \ln(4 \cosh 2J_n) \tag{26}$$

The fixed point is $J_* = 0.6094$ (see Problem 2). $T_c = J/0.6094$. The temperature dependence of the specific heat $C = -T\partial^2 F/\partial^2 T$ is shown on Fig. 3

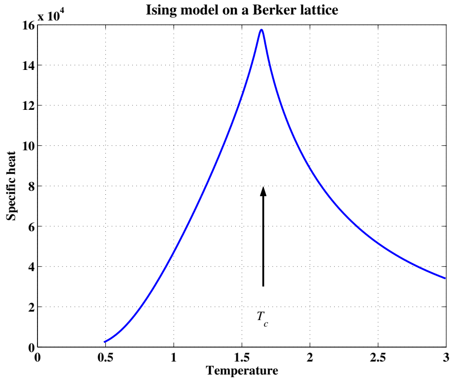

Figure 3: Specific heat for Ising model on Berker lattice.

At $h \neq 0$,

$$Z_\diamondsuit = \sum_{s_1,s_2} e^{hs_1+hs_2} \left(\sum_{s_3} e^{J(s_1+s_2)s_3+hs_3} \right)^2 = Z_0 \sum_{s_1,s_2} e^{J's_1s_2+(h+2\delta h)s_1+(h+2\delta h)s_2} \tag{27}$$

where J' and δh are given by expressions derived in Problem 1. At small h,

$$h' = (1 + \tanh(2J))\, h \tag{28}$$
$$J' = \ln \cosh(2J) \tag{29}$$

At the fixed point $J = J_*$, the magnetic eigenvalue is

$$y_h = \ln(h'/h) = \ln\left(1 + \tanh(2J_*)\right) = \ln\left(\frac{e^{2J_*}}{\cosh 2J_*}\right) = \ln(e^{J_*}) = J_* = 0.6094 \tag{30}$$

The thermal eigenvalue is

$$y_t = \ln(dJ'/dJ) = \ln \frac{2\sinh 2J_*}{\cosh 2J_*} = \ln\left(e^{J_*} - e^{-3J_*}\right) = 0.5180 \tag{31}$$

4. Ising model on a fractal

Consider the Ising problem on a fractal lattice shown in the figure. By using spin blocking, find the renormalization flow for the coupling constant. Consider the fixed points and discuss the ordering types and phase transitions possible in this system.

$$\tilde{Z}_{123} = \sum_{s_{4,5,6}} \exp\left(Ks_4(s_1+s_2) + Ks_5(s_2+s_3) + Ks_6(s_1+s_3) + Ks_4s_6 + Ks_6s_5 + Ks_4s_5\right) \tag{32}$$

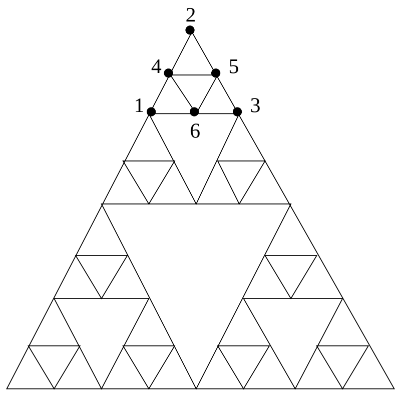

Figure 4: A fractal lattice: Sierpinsky gasket

The new partition function obtained by summing over s_4, s_5, s_6 is

$$Z = Z_0 e^{K'(s_1 s_2 + s_2 s_3 + s_3 s_1)} \tag{33}$$

To find Z_0 and K', consider configurations $s_1 s_2 s_3 = +++$ and $s_1 s_2 s_3 = ++-$. For $s_1 s_2 s_3 = +++$

$$Z_{123} = \sum_{s_4, s_5, s_6} \exp\left(2K(s_4 + s_5 + s_6) + K(s_4 s_5 + s_6 s_5 + s_4 s_6)\right) = e^{9K} + e^{-3K} + 3\left(e^{K} + e^{-3K}\right) \tag{34}$$

and for $s_1 s_2 s_3 = ++-$

$$Z_{123} = \sum_{s_4, s_5, s_6} \exp\left(2K s_4 + K(s_4 s_5 + s_6 s_5 + s_4 s_6)\right) = e^{5K} + 4e^{K} + 3e^{-3K} \tag{35}$$

Hence $Z_0 e^{3K'} = Z_{+++}$ and $Z_0 e^{-K'} = Z_{++-}$. Therefore

$$e^{4K'} = \frac{e^{9K} + e^{-3K} + 3\left(e^{K} + e^{-3K}\right)}{e^{5K} + 4e^{K} + 3e^{-3K}} = e^{4K} \frac{e^{5K} + 3e^{-3K} + 4e^{-7K}}{e^{5K} + 4e^{K} + 3e^{-3K}} \tag{36}$$

and $Z_0^4 = Z_{+++} Z_{++-}^3$. Note that K' is always less than K which means that there is no phase transition in this system. The partition function for the n-level structure

$$Z = \left(\ldots \left(\left(Z_0^3(K) Z_0(K') \right)^3 Z_0(K'') \right)^3 Z_0(K''') \ldots \right)^3 Z_0(K^{(n)}) \tag{37}$$

The specific heat $C = T\partial^2(T \ln Z)/\partial T^2$ is shown on Fig. 5.

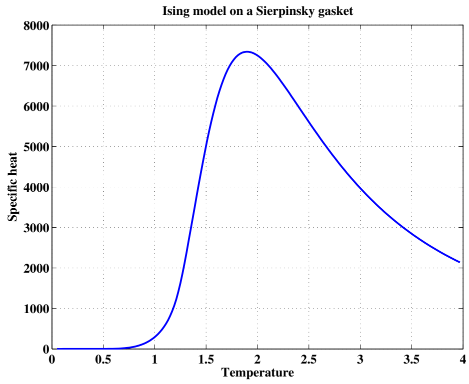

Figure 5: Specific heat for Ising model on a Sierpinsky gasket

5. Relationship between critical indices

Consider the principal theromodynamical indices α, β, γ, δ, η, ϵ, μ defined in Lecture 5. Based on the scaling of the Ising model free energy near RG fixed point and on the correlation length scaling, find the relation in terms of the RG flow eigenvalues y_t and y_h for as many of the indices α, ..., μ, as you can. Show that all of these indices can be expressed in terms of two "independent" indices, say α and β.

The relations between critical exponents $\alpha \ldots \eta$ and RG flow eigenvalues are found in Lecture 7. To find ϵ and μ, we can use scaling argument as follows. Specific heat $C \propto \partial^2 F/\partial\tau^2$ is a function of τ and $h/\tau^{y_h/y_t}$. Since $C \propto \tau^{-\alpha}$ at $h = 0$ and $C \propto h^{-\epsilon}$ at $\tau = 0$, the following relation must hold: $(\tau^{y_h/y_t})^{-\epsilon} = \tau^{-\alpha}$, i.e. $\epsilon = y_t/y_h\alpha = (2y_t - d)/y_h$. Similar argument for correlation length gives $\mu y_h/y_t = \nu$, i.e. $\mu = y_t/y_h\nu = 1/y_h$.

The thermal and magnetic eigenvalues can be expressed in terms of two "independent" indices:

$$y_t = \frac{d}{2-\alpha} \qquad\qquad y_h = \frac{2-\alpha-\beta}{2-\alpha}d \tag{38}$$

hence all the indices can be expressed in terms of α and β.