8.334: Statistical Mechanics II Problem Set # 1 Due: **2/12/03**

Long-range order, symmetry and soft modes

1. Superfluid dynamics

Interacting Bose gas in a superfluid state is described by the Hamiltonian

$$\mathcal{H} = \int \left(\frac{\hbar^2}{2m} |\nabla \psi|^2 + \frac{g}{2} (\bar{\psi}\psi - n)^2 \right) d^3 r \tag{1}$$

with $\psi(r)$ the Bose condensate wavefunction. Here m, g and n are particle mass, interaction, and density.

a) Find the ground state. Compare the symmetry of the Hamiltonian and the symmetry of the ground state. How many soft modes do you expect to have in this system?

The ground state is

$$|\psi| = \sqrt{n} e^{i\theta} \tag{2}$$

where the phase θ is arbitrary. The Hamiltonian (1) is invariant under gauge transformation $\psi \to e^{i\alpha}\psi$. The grounds states do not have this symmetry.

b) The dynamics is described by the Gross-Pitaevsky equation[1]

$$i\hbar \partial_t \psi = \frac{\delta \mathcal{H}}{\delta \bar{\psi}} = -\frac{\hbar^2}{2m} \nabla^2 \psi + g(\bar{\psi}\psi - n)\psi \tag{3}$$

The equation for $\bar{\psi}$ is a complex conjugate of Eq.(3), $-i\hbar\partial_t \bar{\psi} = \delta\mathcal{H}/\delta\psi$. Consider a perturbed ground state $\psi(r,t) = \psi^{(0)} + \delta\psi(r,t)$ and linearize in $\delta\psi(r,t)$ the pair of equations for ψ and $\bar{\psi}$ (Eq.(3) and its conjugate). Find a plane wave solution and characterize soft modes in a superfluid. How does the dispersion relation $\omega(k)$ look at small k?

Consider perturbation $\delta\psi$ of the ground state $\psi = \psi^{(0)} + \delta\psi$. Then the linearized equation for $\delta\psi$ and $\delta\bar{\psi}$ are

$$i\hbar\partial_t \delta\psi = -\frac{\hbar^2}{2m} \nabla^2 \delta\psi + g\left(\psi^2 \delta\bar{\psi} + n\delta\psi\right) \tag{4}$$

$$-i\hbar\partial_t \delta\bar{\psi} = -\frac{\hbar^2}{2m} \nabla^2 \delta\bar{\psi} + g\left(\bar{\psi}^2 \delta\psi + n\delta\bar{\psi}\right) \tag{5}$$

Seek the plain wave solution $\delta\psi \propto \exp(-i\omega t + kr)$. Then the dispersion relation is

$$\det \begin{pmatrix} \hbar\omega - \frac{k^2\hbar^2}{2m} - gn & -gne^{2i\theta} \\ -gne^{-2i\theta} & -\hbar\omega - \frac{k^2\hbar^2}{2m} - gn \end{pmatrix} = 0 \tag{6}$$

The solution is

$$\omega(k) = \sqrt{\left(\frac{k^2}{2m}\right)^2 \hbar^2 + \frac{gn}{m} k^2} \tag{7}$$

[1]This equation is also known as the nonlinear Schrödinger equation and the time-dependent Ginzburg-Landau equation.

For small k we obtain $\omega = ck$ where $c = \sqrt{gn/m}$.

c) Treating the excitations found in part b) as Bose particles, find the entropy and specific heat of the superfluid (1) at low temperature $T \ll gn$.

The thermodynamic potential is

$$\Omega = \frac{1}{\beta} \int \frac{d^3k}{(2\pi)^3} \ln\left(1 - e^{-\beta\hbar\omega}\right) \tag{8}$$

We are concerned with the low-k modes described by the dispersion relation $\omega = ck$. In this case all the dimensional quantities can be scaled out of the integral:

$$\Omega = \frac{4\pi}{\beta^4(\hbar c)^3(2\pi)^3} \int_0^\infty \ln\left(1 - e^{-x}\right) x^2 dx = -\frac{T^4}{6\pi^2(\hbar c)^3} \int_0^\infty \frac{x^3 dx}{e^x - 1} = -\frac{\pi^2 T^4}{90(\hbar c)^3} \tag{9}$$

The entropy is

$$S = -\frac{\partial \Omega}{\partial T} = \frac{2\pi^2 T^3}{45(\hbar c)^3} \tag{10}$$

The specific heat is

$$c = T\frac{\partial S}{\partial T} = \frac{2\pi^2 T^3}{15(\hbar c)^3} \tag{11}$$

2. Spin waves in a Heisenberg ferromagnet

Consider a classical spin model of a ferromagnet

$$\mathcal{H} = -\frac{1}{2} \sum_{r \neq r'} J(r - r')\, \mathbf{S}_r \cdot \mathbf{S}_{r'} \tag{12}$$

Here $\mathbf{S}_r$ is a unit vector and $J(r - r') > 0$ is spin exchange interaction. This model describes a large spin limit of the quantum Heisenberg problem.

a) Find the ground state. Compare the symmetry of the Hamiltonian (12) and the symmetry of the ground state. How many soft modes do you expect to have in this system?

The ground state is $\mathbf{S} = \mathbf{S}^0$ where $\mathbf{S}^{(0)}$ is uniform. The Hamiltonian is invariant under uniform rotations in the spin space, i.e. the degeneracy space is a 2D sphere. It looks like there should be two soft modes, but actually there is only one.

b) The dynamics of each spin is precession in the magnetic field of neighboring spins,

$$\partial_t \mathbf{S}_r = \mathbf{S}_r \times \mathbf{B}_r\,, \qquad \mathbf{B}_r \equiv -\delta\mathcal{H}/\delta\, \mathbf{S}_r = \sum_{r'} J(r - r')\, \mathbf{S}_{r'} \tag{13}$$

Consider a long wavelength spin wave, $\mathbf{S}_r(t) = \mathbf{S}_r^{(0)} + \delta\, \mathbf{S}_r(t)$, linearize Eqs.(13) in $\delta\, \mathbf{S}$, and find a plane wave solution for spin waves. How does the dispersion relation $\omega(k)$ look at small k?

The linearized equation is

$$\partial_t \delta\mathbf{S} = \mathbf{S}_r^{(0)} \times \delta\mathbf{B} + \delta S_r \times \mathbf{B}^{(0)} \tag{14}$$

The perturbation of the magnetic field is

$$\delta\mathbf{B} = \sum_{r'} J(r - r')\delta S_{r'} = \sum_{r'} J(r - r') \left(\delta\mathbf{S}_r + (\mathbf{r} - \mathbf{r}') \cdot \nabla\delta\mathbf{S}_r + \frac{1}{2}((\mathbf{r} - \mathbf{r}') \cdot \nabla)^2 \delta\mathbf{S}_r + \ldots\right) \tag{15}$$

Since

$$\sum_r J(\mathbf{r}-\mathbf{r}')(\mathbf{r}-\mathbf{r}') = 0 \tag{16}$$

$$\sum_r J(\mathbf{r}-\mathbf{r}')(\mathbf{r}-\mathbf{r}')^2 = 6J_0 \tag{17}$$

(see the note after equation (17) in Lecture 1), we obtain the so-called Landau-Lifshits equation

$$\partial_t \delta\mathbf{S} = J_0\, \mathbf{S}^{(0)} \times \nabla^2 \delta\mathbf{S} \tag{18}$$

Choose $\mathbf{S}^{(0)} = \hat{z}$, $\delta\mathbf{S} = \delta S_x \hat{x} + \delta S_y \hat{y}$ and seek a plane wave solution $\delta S \propto \exp(-i\omega t + kr)$. Then the dispersion relation is

$$\det \begin{pmatrix} i\omega & J_0 k^2 \\ -J_0 k^2 & i\omega \end{pmatrix} = 0 \tag{19}$$

or

$$\omega(k) = J_0 k^2 \tag{20}$$

c) Treating spin waves as Bose particles, find the entropy and specific heat of the ferromagnet (12) at low temperature $T \ll J$.

$$\Omega = T \int \ln\left(1 - e^{-\beta\hbar\omega}\right) \frac{d^3k}{(2\pi)^3} = \frac{4\pi T}{(\beta\hbar J_0)^{3/2}(2\pi)^3} \int_0^\infty dx\, x^2 \ln\left(1 - e^{-x^2}\right) \tag{21}$$

The integral is

$$\int_0^\infty dx\, x^2 \ln\left(1 - e^{-x^2}\right) = -\frac{1}{3}\int_0^\infty \frac{y^{3/2}\, dy}{e^y - 1} = -\frac{1}{3}\Gamma(5/2)\zeta(5/2) \tag{22}$$

Therefore

$$\Omega = -\frac{\Gamma(5/2)\zeta(5/2)}{6\pi^2(\hbar J_0)^{3/2}} T^{5/2} \tag{23}$$

$$S = \frac{5\Gamma(5/2)\zeta(5/2)}{12\pi^2(\hbar J_0)^{3/2}} T^{3/2} \tag{24}$$

$$c = \frac{5\Gamma(5/2)\zeta(5/2)}{8\pi^2(\hbar J_0)^{3/2}} T^{3/2} \tag{25}$$

d) Spin system in external magnetic field is described by $\mathcal{H} \to \mathcal{H} - \sum_r \mu\, \mathbf{S}_r \cdot \mathbf{B}_{ext}$ with μ the spin magnetic moment (Bohr's magneton for electron spin). Consider once more the above questions a), b) and c). Comment on the difference with the $B_{ext} = 0$ case.

a') In magnetic field the ground state is $\mathbf{S}_r = \mathbf{S}^{(0)}$ where $\mathbf{S}^{(0)} \parallel \mathbf{B}$. There is no degeneracy.

b') The linearized equation of motion now is

$$\partial_t \delta\mathbf{S} = \mathbf{S}_r^{(0)} \times \delta\mathbf{B} + \delta S_r \times \mathbf{B}_{ext} = J_0\, \mathbf{S}^{(0)} \times \nabla^2\delta\mathbf{S} - \mathbf{B}_{ext} \times \delta\mathbf{S} \tag{26}$$

Choose $\mathbf{B}_{ext}$, $\mathbf{S}^{(0)} \parallel \hat{z}$ and look for plane wave solutions. Then

$$\det \begin{pmatrix} i\omega & J_0k^2 + |B_{ext}| \\ -(J_0k^2 + |B_{ext}|) & i\omega \end{pmatrix} = 0 \tag{27}$$

or

$$\omega(k) = \omega_0 + J_0 k^2 \tag{28}$$

where $\omega_0 = |B_{ext}|$. The dispersion relation has a cut-off at $\omega = \omega_0$. There are no low-frequency modes.

c')

$$\Omega = T \int \ln\left(1 - e^{-\beta\hbar\omega}\right) \frac{d^3k}{(2\pi)^3} = -\frac{\Gamma(5/2)}{6\pi^2(\hbar J_0)^{3/2}} T^{5/2} f_{5/2}(e^{-\lambda}) \tag{29}$$

where

$$\lambda = \hbar\beta\omega_0 \tag{30}$$

$$f_m(z) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{dx\, x^{m-1}}{z^{-1}e^x - 1} = \sum_{j=1}^\infty \frac{z^j}{j^m} \tag{31}$$

At high temperatures, $T \gg \hbar\omega_0$ the thermodynamic Al properties are the same as at $B_{ext} = 0$, while at low temperatures, $T \ll \hbar\omega_0$, there are almost no thermally excited modes and the thermodynamic quantities are exponentially small

$$\Omega \propto T^{5/2} e^{-\hbar\omega_0/T} \tag{32}$$

$$S \propto \omega_0 T^{1/2} e^{-\hbar\omega_0/T} \tag{33}$$

$$c \propto \omega_0^2 T^{-1/2} e^{-\hbar\omega_0/T} \tag{34}$$

$$\tag{35}$$