Field-theoretic RG (continued).

1. RG for magnetic coupling. Consider Wilson's RG scheme in $d = 4 - \epsilon$ near critical point in the presence of a weak magnetic field

$$\mathcal{H} = \int \left[\frac{1}{2} \left(\tau \mathbf{m}^2 + K(\nabla \mathbf{m})^2 \right) + u(\mathbf{m}^2)^2 - \mathbf{h} \cdot \mathbf{m} \right] d^d x \tag{1}$$

(here $\mathbf{m}$ is an n-component field). We would like to extend the RG analysis to find scaling transformation of the $\mathbf{h} \cdot \mathbf{m}$ term.

a) Consider the gaussian problem with $u = 0$. Show that, since the term in question is linear, it does not change under coarse-graining. (Set $\mathbf{m} = \mathbf{m}_< + \mathbf{m}_>$ and average over $\mathbf{m}_>$ in the partition function.) Find how this term is changed after rescaling $\mathbf{q} = b^{-1}\mathbf{q}'$ and field renormalization $\mathbf{m} = z\mathbf{m}'$ with z chosen to maintain the rigidity K fixed. Put the recursion relation for h in a differential form

$$dh/dl = y_h^{(0)} h\,, \quad l = \ln(\Lambda/\Lambda') \tag{2}$$

b) At small u, perform cummulant expansion in $\ln(\mathrm{tr}_> e^{-\mathcal{H}_{\mathrm{int}}})$ up to second order in u. Let us focus on the recursion relation for rigidity (Eq.26, Lecture 12),

$$\tilde{K} = K - u^2 A''(\mathbf{q} = 0) \tag{3}$$

where A'' is obtained from contributions of the form $\mathbf{m}_<^2$, as described in Lecture 12. (Fig.2, Lecture 12). Find the field renormalization factor z, taking into account this correction to K. Derive a recursion relation for $\mathbf{h}$ correct to the second order in u^2. Assuming that $A''(\mathbf{q} = 0) = C \ln(\Lambda/\Lambda')$ (as will be found below), what is the correction to the field scaling exponent $y_h^{(0)}$?

c) Now we face the problem of calculating the correction to rigidity. Note that all diagrams in Fig.2 (Lecture 12), except the last one, give *local* contributions $\int \mathbf{m}_<^2 d^d x$, and contribute only to τ, but not to K. The last diagram in Fig.2 (Lecture 12), and several other similar diagrams with different combinatorial factors (*draw these diagrams and determine the combinatorial factors!*) gives an expression

$$\delta\mathcal{H} = -\frac{1}{2} u^2 \sum_{\mathbf{q}} |\mathbf{m}_{<,\mathbf{q}}|^2 A(\mathbf{q}) \tag{4}$$

which can be expanded in $\mathbf{q}$ to obtain a contribution to $A''(0)$. Technically, the analysis of the diagram is much simplified by going to position representation,

$$\delta\mathcal{H} = -\frac{1}{2} u^2 \iint \mathbf{m}_{<,i}(\mathbf{x}) \mathbf{m}_{<,i}(\mathbf{x}') A(\mathbf{x} - \mathbf{x}') d^d x d^d x' \tag{5}$$

$$A(\mathbf{x} - \mathbf{x}') = f(n)\, G^3(\mathbf{x} - \mathbf{x}')\,, \quad G(\mathbf{x}) = \int \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{\tau + K\mathbf{q}^2} \frac{d^d q}{(2\pi)^d} \tag{6}$$

where $f(n)$ is a combinatorial factor.

Evaluate $G(\mathbf{x})$ at $d = 4$ and $\tau = 0$, and *show that* $G(\mathbf{x}) = 1/(4\pi^2|\mathbf{x}|^2)$. Using this result, write

$$A(\mathbf{q}) = f(n) \int e^{-i\mathbf{q}\cdot\mathbf{x}} G^3(\mathbf{x}-\mathbf{x}') d^4x \tag{7}$$

expand in $\mathbf{q}$ and calculate $A''(\mathbf{q}=0)$. (Here you'll have to think how to handle a logarithmic divergence using Λ and Λ').

d) Finally, combine your answer for A'' with the result of part b) and calculate the $O(\epsilon^2)$ correction to the magnetic field scaling exponent y_h.

Find the correlation function scaling exponent η, and the susceptibility exponent γ (Lectures 5, 7) corrected by ϵ-expansion to lowest order in ϵ.

2. Asymptotic symmetry. The symmetry of a system at a critical point can be higher than away from it. The higher symmetry arises due to critical fluctuations that suppress nonsymmetric (anisotropic) interactions.

a) One example of this phenomenon is provided by the problem with cubic anisotropy from the last homework,

$$\mathcal{H} = \int \left[\frac{1}{2}\left(\tau\mathbf{m}^2 + K(\nabla\mathbf{m})^2\right) + u(\mathbf{m}^2)^2 + v\sum_{i=1}^{n} m_i^4 \right] d^dx \tag{8}$$

Using RG equations derived earlier for this problem, show that at $n < 4$ the only stable fixed point is characterized by zero anisotropy, $v = 0$.

This means that even if the anisotropy v is present microscopically, it is suppressed by fluctuations and rotational symmetry is restored at the critical point. Experimentally, asymptotic symmetry is manifest in the values of critical indices, which are determined by asymptotic symmetry, rather than the symmetry of the Landau hamiltonian. For the above example of an n-component field, the indices differ for the Ising class (describing anisotropic system) and the XY and Heisenberg classes.

b) Another classic example of asymptotic symmetry is intersection of two second order transition lines. Near the intersection point, the hamiltonian can be written as

$$\mathcal{H} = \int \left[\frac{1}{2}\left(\tau_1\phi_1^2 + K_1(\nabla\phi_1)^2 + u_1(\phi_1^2)^2\right) + \frac{1}{2}\left(\tau_2\phi_2^2 + K_2(\nabla\phi_2)^2 + u_2(\phi_2^2)^2\right) + v\phi_1^2\phi_2^2 \right] d^dx \tag{9}$$

At the critical point $\tau_{1,2} = 0$, derive the RG flow equations for the couplings u_1, u_2, and v, using perturbation theory up to second order in interaction. Analyze stability of the nontrivial fixed point with $v = 2u_1 = 2u_2$, at which the ϕ^4 terms combine into

$$\frac{v}{2}\left(\phi_1^2 + \phi_2^2\right)^2 \tag{10}$$

i.e. the problem possesses rotational symmetry O_{m+n}. Under what conditions this fixed point it stable?