Field-theoretic RG. Wilson-Fisher fixed point.

1. Wick theorem. Consider a real-valued n-component field $\eta_i(\mathbf{x})$ with gaussian probability distribution

$$P(\eta) \propto \exp(-\mathcal{H})\,, \quad \mathcal{H}(\phi) = \int \frac{1}{2}\left(\tau\eta^2 + K(\nabla\eta)^2\right) d^d x \tag{1}$$

We are interested in statistics of Fourier components of this field,

$$\eta_i(\mathbf{x}) = \int \eta_i(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x}} d^d x \tag{2}$$

a) Show that

$$\langle \eta_i(\mathbf{q})\rangle = 0\,, \quad \langle \eta_i(\mathbf{q})\eta_{i'}(\mathbf{q}')\rangle = \frac{\delta_{ii'}\delta(\mathbf{q}+\mathbf{q}')}{\tau + K\mathbf{q}^2}\,, \tag{3}$$

b) Show that

$$\langle \eta_i(\mathbf{q})\eta_{i'}(\mathbf{q}')\eta_{i''}(\mathbf{q}'')\rangle = 0\,, \quad \langle \eta_{i_1}(\mathbf{q}_1)\eta_{i_2}(\mathbf{q}_2)\eta_{i_3}(\mathbf{q}_3)\eta_{i_4}(\mathbf{q}_4)\rangle = \langle \eta_{i_1}(\mathbf{q}_1)\eta_{i_2}(\mathbf{q}_2)\rangle\langle \eta_{i_3}(\mathbf{q}_3)\eta_{i_4}(\mathbf{q}_4)\rangle$$
$$+\langle \eta_{i_1}(\mathbf{q}_1)\eta_{i_3}(\mathbf{q}_3)\rangle\langle \eta_{i_2}(\mathbf{q}_2)\eta_{i_4}(\mathbf{q}_4)\rangle + \langle \eta_{i_1}(\mathbf{q}_1)\eta_{i_4}(\mathbf{q}_4)\rangle\langle \eta_{i_3}(\mathbf{q}_3)\eta_{i_2}(\mathbf{q}_2)\rangle \tag{4}$$

c) Generalize the results for an average of an arbitrary product $\langle \eta_{i_1}(\mathbf{q}_1)...\eta_{i_n}(\mathbf{q}_n)\rangle$.

2. Field statistics. Cummulants. Consider a partition function for a field $\phi(\mathbf{x})$ defined by a Hamiltonian

$$\mathcal{H}(\phi) = \int F(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi, ...) d^d x \tag{5}$$

where F is a polynomial in the field and its gradients. (For example, $F = \frac{1}{2}\left(\tau\phi^2 + K(\nabla\phi)^2\right) +$ $g\phi^4$ in Landau theory.) The Hamiltonian $\mathcal{H}$ defines a probability distribution $P \propto e^{-\beta\mathcal{H}}$ of fields and can be used to find an average of any quantity expressed in terms of $\phi(\mathbf{x})$.

Let us consider an average of an exponential of an extensive quantity of the form

$$P_V(\lambda) = \langle \exp\left(\lambda G(\phi)\right)\rangle_{\mathcal{H}}\,, \quad G = \int U(\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi, ...) d^d x \tag{6}$$

where λ is a parameter, U is a polynomial and the integral over $\mathbf{x}$ is taken over a domain of a very large volume V.

a) Show that, if correlations of ϕ are finite range or decrease with distance sufficiently rapidly, the quantity $P_V(\lambda)$ is exponential in V. (Hint: Divide volume V into V_1 and V_2, so that $V_1 + V_2 = V$, and think of a relation between $P_V(\lambda)$, $P_{V_1}(\lambda)$, and $P_{V_2}(\lambda)$.)

b) Consider an expansion of $\ln P_V(\lambda)$ in powers of λ and show that

$$\ln P_V(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n}\lambda^n \langle\langle G^n\rangle\rangle \tag{7}$$

where all *cummulants* $\langle\langle G^n\rangle\rangle$, also known as *irreducible moments*, are proportional to system volume V.

c) By expanding the exponent in Eq.(6) and the logarithm in Eq.(7), relate the first four cummulants with the moments $\langle G^n\rangle$ by showing that

$$\langle\langle G\rangle\rangle = \langle G\rangle\,, \quad \langle\langle G^2\rangle\rangle = \langle\delta G^2\rangle\,, \quad \langle\langle G^3\rangle\rangle = \langle\delta G^3\rangle\,, \quad \langle\langle G^4\rangle\rangle = \langle\delta G^4\rangle - 3\langle\delta G^2\rangle^2 \qquad (8)$$

where $\delta G = G - \langle G\rangle$. One can view cummulants as generalized moments of the distribution with a nontrivial property that the cummulants of all orders are proportional to system volume, $\langle\langle G^n\rangle\rangle \propto V$, while the moments $\langle G^n\rangle \propto V^n$

3. Cubic anisotropy. Consider Landau Hamiltonian for an n-component field $\mathbf{m}$ with a quartic anisotropy term:

$$\mathcal{H} = \int \left[\frac{1}{2}\left(\tau\mathbf{m}^2 + K(\nabla\mathbf{m})^2\right) + u((\nabla\mathbf{m})^2)^2 + v\sum_{i=1}^{n} m_i^4\right] d^dx \qquad (9)$$

This problem with $n = 3$ describes a critical point of Heisenberg ferromagnet in a cubic crystal with a weak spin-orbital coupling of strength v that tends to align spins with crystal axes.

Following the steps that led us in class to the Wilson-Fisher fixed point, derive RG flow equations for the two couplings:

$$\frac{du}{dl} = \epsilon u - 4C\left[(n+8)u^2 + 6uv\right]\,, \quad \frac{dv}{dl} = \epsilon v - 4C\left[12uv + 9v^2\right] \qquad (10)$$

where $C = K_d\Lambda^d/(\tau + K\Lambda^2)^2 \approx K_4/K^2$ is a constant.

Analyze the RG flow in the $u - v$ plane. Find fixed points for $n < 4$ and $n > 4$, and discuss relevance of the cubic anisotropy at the critical point in each case.