Fluctuations in low space dimension.

1. 1D Ising model, correlation length, susceptibility

a) Consider a pair spin correlation function in the 1D Ising model $\mathcal{H} = \sum_i K s_i s_{i+1}$, $s_i = \pm 1$. This problem can be solved by the transfer matrix method, discussed in recitation, or by the method of auxiliary link variables $\xi_i = s_i s_{i+1}$, discussed in Lecture 6 [Eqs.(10,11,12)]. Show that in terms of the link variables, the correlation function is

$$\langle s_{i_1} s_{i_2} \rangle = \langle \prod_{i_1 \le j < i_2} \xi_j \rangle \,, \quad i_1 < i_2. \tag{1}$$

Using the hamiltonian for ξ's (Eq.(11), Lec. 6), average the product (1). Comparing to the asymptotic formula $C_2(x - x') \propto \exp(-|x - x'|/\xi)$, determine the correlation length ξ as a function of temperature.

b) At distances much larger than ξ the system is disordered, while at distances less than ξ the spins are strongly correlated. In RG language, for block sizes $\ge \xi$ the block spins are effectively decoupled from each other. (Indeed, renormalized temperature becomes very large at the length scale ξ, indicating decoupling.)

Consider spin susceptibility $\chi = dm/dh|_{h=0}$. Treating the problem as a system of independent block spins of size ξ, each taking values $\pm\xi$, show that $\chi \sim \xi/T$.

c) It is interesting to look at the susceptibility in the ξ variables representation. Start with the general expression $\chi = d^2 \ln Z/dh^2|_{h=0}$, and write the susceptibility in terms of ξ_i. Average over ξ's using the method of part a) and compare the result with the above estimate for independent block spins.

2. Magnetization and susceptibility of a 2D Heisenberg ferromagnet

a) Use the thermodynamic potential of spin excitations of a 2D Heisenberg ferromagnet at low temperature (Problem 2, PS#1), and find magnetization $m = d \ln Z/dh$ as a function of magnetic field h. Show that the zero temperature value $m(0)$ is reduced by thermal fluctuations so that the difference $\delta m = m(0) - m(T)$ diverges logarithmically as $h \to 0$.

b) Relate the divergence found in part a) with the logarithmic divergence from the Mermin and Wagner theorem. The latter is concerned with transverse magnetization fluctuations, so one has to connect the depletion of spin polarization along the field with its transverse fluctuations.

c) It is interesting to consider the zero field susceptibility of this system. From RG analysis of the nonlinear sigma model, we know that the effective coupling becomes very small at some temperature-dependent length scale $\xi(T)$. Use the arguments similar to those in part b) of Problem 1 to estimate the susceptibility. Does your estimate agree with the above result for $m(T)$ extrapolated to $h = 0$?

3. The sine-Gordon model

Consider a 2D gaussian problem of a scalar field $\phi(\mathbf{x})$ perturbed by a small cos term:

$$\mathcal{H} = \iint \left(\frac{1}{2}(\partial_\mu \phi)^2 + \lambda \cos(\beta\phi) \right) d^2x \tag{2}$$

Here λ is a coupling constant, and β is a dimensionless parameter.

Apply the field-theoretic RG transformation described in Lecture 8 (coarse grain + rescale + renormalize hamiltonian) to this problem, treating the cos term as a perturbation. On each RG step, split the field ϕ into the fast and slow component, and average the Hamiltonian[1] (2) over the fast fluctuations, assuming them to be described by the quadratic part of (2).

a) Show that the hamiltonian form is preserved upon RG and derive the RG flow equation, first order in a weak coupling constant:

$$d\lambda/dt = -f(\beta)\,\lambda\,, \quad t \equiv \ln(\Lambda/\Lambda') \tag{3}$$

Find the function $f(\beta)$ and show that, depending on the value of β, the coupling λ either decreases, or increases. For λ decreasing, the problem turns into a freely fluctuating gaussian field, corresponding to a disordered state, whereas for λ increasing, the fluctuations freeze at a certain length scale (estimate it), corresponding to the ordered state.

b) Add a perturbation $\mu\cos(\beta'\phi)$ to the problem (2). Show that this perturbation is irrelevant near the phase transition found in part a) when $\beta' > \beta$.

On the contrary, a perturbation with $\beta' < \beta$ is relevant. How does it affect the phase transition?

4. Magnetization in the XY problem.

Consider an XY model for planar ferromagnet in a weak in-plane magnetic field. The magnetization is described by a 2-component unit vector $\mathbf{n} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}$.

a) Show that the hamiltonian, written in terms of the angular variable θ, has the form

$$\mathcal{H} = \iint \left(\frac{1}{2}J(\partial_\mu\theta)^2 - h\cos\theta\right) d^2x \tag{4}$$

(here the field h is applied along the x axis). The hamiltonian is formally identical to the sine-Gordon problem (2), except that in the XY case the variable θ lives on a circle, not on a line. For the moment, however, let us ignore this difference.

b) The magnetization induced by the field h is given by $\bar{m} = \langle\cos\theta\rangle$. Consider the limit of a weak field, when thermal fluctuations suppress magnetization. Assume that the temperature is below the Kosterlitz-Thouless temperature, so that vortices can be excluded from the analysis.

To find the magnetization, go back to the RG analysis of Problem 3 a). At low temperature, when $f(\beta) < 0$ and coupling grows under renormalization, determine the length scale Λ_* at which the second term of the hamiltonian (4) reaches the first term. Argue that at larger length scales the fluctuations freeze out and the system becomes ordered. By averaging $\cos\theta$ over the fluctuations with wavelengths up to Λ_*, find the magnetization as a function of magnetic field.

[1] It would be more appropriate to average the exponent $e^{-\mathcal{H}}$, which is a slightly more difficult task. However, to the first order in λ the result is the same, so averaging the Hamiltonian in this problem is sufficient. The reason for this, as we discuss later, is that "the first loop RG approximation" is nonzero, and so the transition arises already in the lowest order of perturbation theory. Wait till PS#7 to see an example of RG based on averaging $e^{-\mathcal{H}}$.