**Real space renormalization. Scaling theory.**

**1. Renormalization for 1D Ising model**

Consider a 1D Ising model with the nearest neighbor interaction in external magnetic field

$$\mathcal{H} = -\sum_i \left(J s_i s_{i+1} + h s_i\right) \tag{1}$$

To apply the renormalization group to this problem, generalize the decimation procedure discussed in class to include the coupling to magnetic field. Obtain a renormalization mapping

$$J' = f_1(J,h)\,, \quad h' = f_2(J,h) \tag{2}$$

Linearize the mapping in $h$ in order to study the renormalization flow at small $h$.

Draw schematically the flow in the $(J,h)$ plane. What are the fixed points of this flow? Which of them are stable or unstable? Which states correspond to the fixed points?

**2. Migdal-Kadanoff renormalization for Ising model**

a) Apply real space renormalization to Ising ferromagnet in magnetic field. Use the bond-moving algorithm, as discussed in class. Sketch the RG flow on the $(J,h)$ plane, identify fixed points, stable and unstable, and discuss their physical meaning.

b) Calcualte the temperature and magnetic field eigenvalues $y_t$, $y_h$ of the RG flow at the fixed point found in part a). Assuming scaling of the singular part of the free energy,

$$f(\tau,h) = b^{-d} f(b^{y_t}\tau, b^{y_h}h) \tag{3}$$

with small $h$, $\tau$, and $b=2$ the rescaling factor, predict the form of singularity in the specific heat and susceptibility. Compare to the exact results (Eq.2, Lecture 5).

**3. Exact renormalization on a hierarchical lattice**

Renormalization group theory of a phase transition aren't necessarily aproximate. Consider an Ising model on a hierarchical lattice shown in Fig. and apply to it the decimation RG, as in problem 1. Find the RG transformation, idenify its fixed points, and sketch the flow on the $(J,h)$ plane.

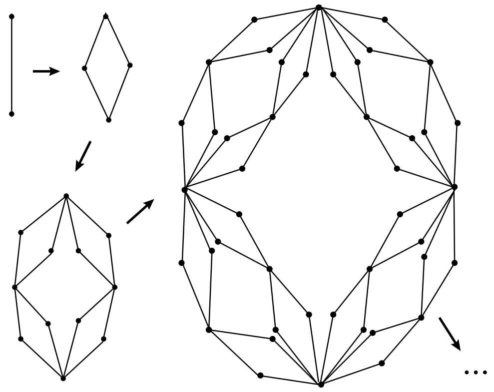


Figure 1: Hierarchical Berker lattice for which decimation can be carried out exactly.

**4. Ising model on a fractal**

Consider the Ising problem on a fractal lattice shown in the figure. By using spin blocking, find the renormalization flow for the coupling constant. Consider the fixed points and discuss the ordering types and phase transitions possible in this system.

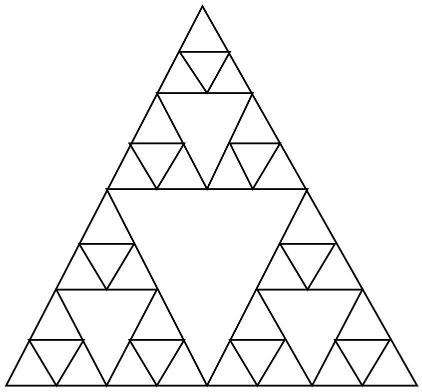

Figure 2: A fractal lattice: Sierpinsky gasket

**5. Relationship between critical indices**

Consider the principal theromodynamical indices $\alpha$, $\beta$, $\gamma$, $\delta$, $\eta$, $\epsilon$, $\mu$ defined in Lecture 5. Based on the scaling of the Ising model free energy near RG fixed point and on the correlation length scaling, find the relation in terms of the RG flow eigenvalues $y_t$ and $y_h$ for as many of the indices $\alpha$, ..., $\mu$, as you can. Show that all of these indices can be expressed in terms of two “independent” indices, say $\alpha$ and $\beta$.