The Ising model and beyond.

1. Duality in 2D Ising model. There is a special kind of transformations of statistical problems known as duality relations, that allow to map one problem to another problem and back. (One familiar example of duality is the relation between the XY problem and the Coulomb gas considered in Lecture 10.) Transformations of that sort are usually quite useful. If a duality transformation is discovered, even if it does not allow to solve the problem, by comparing the problem with its dual one can extract some nontrivial information. Here, by constructing a duality transformation for 2D Ising model, you will obtain an exact answer for the transition temperature.

a) Consider Ising model on a square lattice:

$$\mathcal{H} = -J \sum_{\langle i\, i' \rangle} s_i s_{i'} \tag{1}$$

with the sum taken over pairs of nearest neighbors. Since $s_i = \pm 1$, one can rewrite the Hamiltonian in terms of link variables $\phi_{i\,i'}$ define on the bonds connecting sites i and i' as

$$\mathcal{H} = \frac{J}{2} \sum_{\langle i\, i' \rangle} \phi_{i\,i'}^2 + \text{const}\,, \quad \phi_{i\,i'} = s_i - s_{i'} \tag{2}$$

The variables $\phi_{i\,i'}$ are not independent. Show that they satisfy a constraint $\sum_{\text{plaquette}} \phi = 0$ for each lattice plaquette.

b) To construct duality transformation, replace the sum tr_s in the Ising partition function by a summation tr_ϕ over all link variables with a constraint for each plaquette

$$Z = \mathrm{tr}_\phi \left[e^{-\beta \mathcal{H}(\phi)} \prod_{\text{plaquette}} \delta\left(\sum_{\text{plaquette}} \phi \right) \right] \tag{3}$$

Show that

$$\sum_{\eta=\pm 1} \exp\left(\pi i \eta \left(\sum_{\text{plaquette}} \phi \right) \right) = \begin{cases} 1, & \sum_{\text{plaquette}} \phi = 0 \\ 0, & \sum_{\text{plaquette}} \phi \neq 0 \end{cases} \tag{4}$$

Thus for each plaquette the constraints $\prod_{\text{plaquette}} \delta\left(\sum \phi\right)$ can be put in the form

$$\mathrm{tr}_\eta \exp\left(\pi i \sum_\lambda \eta_\lambda \left(\sum \phi_\lambda \right) \right) \tag{5}$$

where $\eta_\lambda = \pm 1$ are Ising variables on a dual lattice, i.e. at the centers of plaquettes (see Fig.1).

c) By inserting the constraint obtained in part b) in the partition function (2) perform summation over link variables $\phi_{i\,i'}$ and obtain a partition function for the dual lattice. Show that in terms of the variables η_λ the partition function has the form of an Ising model with a new coupling constant $\widetilde{J}$. Derive the Kramers-Wannier relation

$$\sinh(2\beta J) \sinh(2\beta \widetilde{J}) = 1 \tag{6}$$

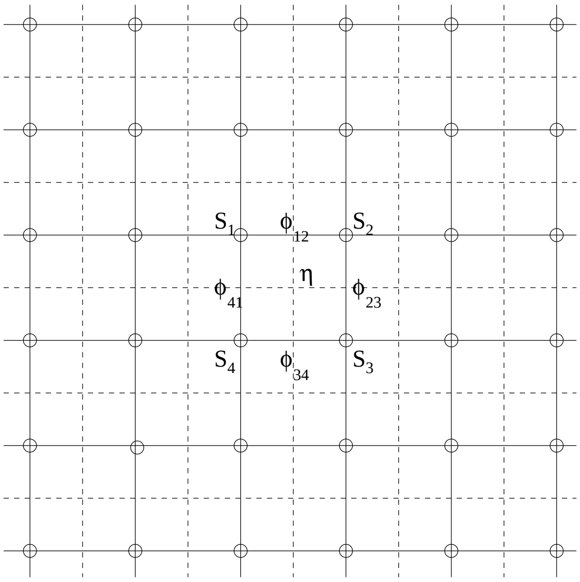

Figure 1: A square lattice with Ising variables $s_i = \pm 1$ at the nodes. Link and plaquette variables are shown.

Note that if one starts with the Ising problem for η's and applies the duality transformation once more, one gets back the initial coupling constant value.

d) Argue that when the system is ordered with respect to variables s_i it is disordered with respect to the dual variables η_λ, and vice versa[1]. Thus the duality transformation maps the low temperature phase onto the high temperature phase, and back. Use this to predict the transition temperature.

e) [*challenging*] Generalize the analysis of parts a)–d) to the anisotropic Ising model on a square lattice with two different couplings J_x and J_y in the x and y directions. Derive the analog of Eq.(6) and use it to determine the transition line in the (J_x, J_y) plane.

2. Ising model on a triangular lattice.

a) Consider the Ising problem (1) on a triangular lattice, and construct a duality transformation. Show that the dual model is the Ising problem on hexagonal (honeycomb) lattice with sites at the centers of triangles. Find a relation between critical temperatures of the Ising problems on the triangular and hexagonal lattices.

In contrast to the square lattice case, the duality transformation alone does not allow to determine the critical temperature for the triangular Ising problem. For that, one must use an additional trick described below.

b) Consider a decimation of the hexagonal Ising problem, similar to that used in the real-space renormalization transformation. Note that the sites of the hexagonal lattice can be divided into two sublattices A and B, so that each A site is surrounded by three B sites, and vice versa. Show that a summation over spins on the A sites performed in the

[1] The dual variables η_λ are also known as *disorder variables*, to emphasize that the nonzero expectation value $\langle \eta \rangle \neq 0$ is found in the disordered phase, when $\langle \eta \rangle = 0$.

partition function, with spins on the B sites fixed, generates a triangular Ising problem for the B sites. How is the coupling constant of the this problem related to the coupling on the hexagonal lattice?

c) Combine the duality transformation of part a) with the decimation procedure of part b) to find the critical temperature of the triangular Ising problem.

3. Scaling around us. Look at the list of fundamental constants below and count how many constants have 1 as the first decimal digit.

Planck constant $\hbar = 1.054 \times 10^{-27}\mathrm{erg\,s}$;
Light speed $c = 2.997 \times 10^{10}\mathrm{cm/s}$;
Electron charge $e = 4.802 \times 10^{-10}\mathrm{esu}$;
Gravitational constant $G = 6.672 \times 10^{-8}\mathrm{cm^3\,g^{-1}\,s^{-2}}$;
Fine structure constant $\alpha = e^2/\hbar c = 7.297 \times 10^{-3}$;
Avogadro number $N_0 = 6.022 \times 10^{23}\mathrm{mole}^{-1}$;
Boltzmann constant $k_B = 1.38 \times 10^{-16}\mathrm{erg/^oK}$;
Universal gas constant $R = N_0 k_B = 8.314 \times 10^7\mathrm{erg/(^oK\,mole)}$;
Electron mass $m = 9.109 \times 10^{-28}\mathrm{g}$;
Atomic mass unit $1.660 \times 10^{-24}\mathrm{g}$;
Proton mass $M_p = 1.672 \times 10^{-24}\mathrm{g}$;
Compton wavelength $\lambda_e = h/mc = 2.426 \times 10^{-10}\mathrm{cm}$;
Bohr's radius $a_0 = \hbar^2/me^2 = 5.291 \times 10^{-9}\mathrm{cm}$;
"Classical electron radius" $r_e = e^2/mc^2 = 2.817 \times 10^{-13}\mathrm{cm}$;
Hydrogen ionization potential $R_\infty = \frac{1}{2}\alpha^2 mc^2 = 13.605\,\mathrm{eV}$;
Rydberg constant $Ry_\infty = \alpha/4\pi a_0 = R_\infty/hc = 1.097 \times 10^5\mathrm{cm}^{-1}$;
Bohr magneton $\mu_B = e\hbar/2mc = 9.273 \times 10^{-21}\mathrm{erg\,g\,s^{-1}}$;
Temperature corresponding to $1\,\mathrm{eV} = 1.16 \times 10^4\,\mathrm{^oK}$

The numbers that start with 1 constitute about 1/3 of all constants. The next largest group are the numbers that start with 2, and so on. If you suspect that it is a coincidence, or a trick, look up other constants, e.g. masses of elementary particles, etc.

Using the ideas of scaling, explain why the first digits 1, 2, 3, ... 9, do not appear with equal probabilities. Find the occurrence probabilities p_n for different digits and compare with your observations.