Quantum phase transitions.

1. Tunneling and 1D Ising model. Consider a quantum particle of mass m in a double well potential,

$$\mathcal{H} = -\frac{\hbar^2}{2m}\partial^2 + U(x)\,, \quad U(x) = u\left((x/a)^2 - 1\right)^2 \tag{1}$$

In the path integral formulation of quantum mechanics, the transition amplitude for the time $(-iT)$ is equal to the classical partition function of a string of length $L = T$ subject to an external potential,

$$Z \sim \int \mathcal{D}\{x(t)\} e^{-\beta S[x(t)]}\,, \quad S[x(t)] = \int_0^L \left(\frac{m}{2}\dot{x}^2 + U(x(t))\right) dt \tag{2}$$

computed at the value of $\beta = 1/\hbar$.

Consider the problem in a quasiclassical limit, when tunneling through the barrier is described by the WKB theory (i.e. by the least action classical path). Show that the saddle point paths can be described as dilute gas of kinks, or instantons, with particle spending a long time in one of the two minima of $U(x)$, then jumping to the other minimum, then back, etc. Establish correspondence with the 1D Ising model (Problem 2, PS#10). What is the statistical mechanical interpretation of the time variable, space variable, the WKB tunneling probability, the even-odd level splitting? What is the quantum mechanical manifestation of the absence of phase transition in the 1D Ising model?

2. String localization and 2D Ising model. A quantum mechanical system with macroscopic behavior described by 2D Ising model can be constructed as follows. Consider a family of particles labeled by integers $-\infty < n < +\infty$, each moving in a double well potential of part a). The particles interact with each other by the nearest neighbor interaction

$$V[..,,x_{-1},x_0,x_1,x_2,...] = \sum_n \frac{k}{2}(x_n - x_{n+1})^2 \tag{3}$$

so that the total quantum mechanical Hamiltonian is a sum of single particle hamiltonians (1) and interparticle interaction:

$$\mathcal{H} = \sum_n \mathcal{H}(x_n) + V[x_i] \tag{4}$$

The physics here is that each particle tunnels back and forth between two minima of the potential $U(x)$, while the interaction $\frac{k}{2}(x_n - x_{n+1})^2$ is trying to keep particles close to each other, i.e. prevent them from tunneling independently (see Fig.1).

Argue that in this system two different states are possible: (i) an ordered state, with all particles occupying the same minimum of $U(x)$, thereby breaking the $x \to -x$ symmetry of the problem, and (ii) a disordered state with no correlations between positions of distant particles.

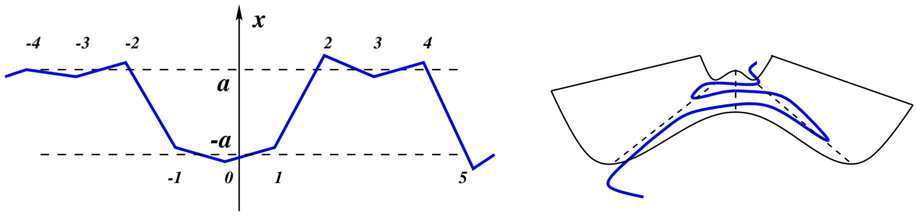

Figure 1: A quantum string in a double well potential.

Using a mean field approach, estimate the critical value of coupling k at which ordering takes place. Argue that the properties near the critical point are described by the 2D Ising model universality class.

This model describes a variety of systems, e.g. a string (polymer chain) in a slit between two parallel surfaces, with equal binding potentials at both surfaces. The string can be pinned to one of the surfaces, or delocalized between the surfaces due to quantum fluctuations.

3. Quantum XY model. Consider a superfluid described by condensate wavefunction $\psi(\mathbf{r}, t)$ with an action

$$S(\psi, \bar{\psi}) = \iint \left(\frac{\hbar}{2} \left(i\bar{\psi}\partial_t\psi - i\psi\partial_t\bar{\psi} \right) - \frac{\hbar^2}{2m}\nabla\bar{\psi}\nabla\psi - U(\psi) \right) d^3r dt \,, \quad U(\psi) = \frac{g}{2}\left(\bar{\psi}\psi - n\right)^2 \tag{5}$$

a) Show that the least action principle, applied to this action functional, gives the dynamical equation for $\psi(\mathbf{r}, t)$ studied in PS#1, Problem 1. (Treat ψ and $\bar{\psi}$ as independent variables!)

b) The low energy field configurations are characterized by nearly constant amplitude $\sqrt{n}$ and fluctuating phase: $\psi = \sqrt{n}\exp(i\theta(\mathbf{r}, t))$. Using this expression and the action (5), derive an effective action for phase fluctuations of the form

$$S(\theta(\mathbf{r}, t)) = \iint \left(\frac{1}{2} a\dot{\theta}^2 - \frac{1}{2} b(\nabla\theta)^2 \right) d^3r dt \tag{6}$$

and find the constants a, b. The result looks like a free field problem, but because of the topologically nontrivial character of the phase parameter space (circle!), this is in fact an XY problem.

c) To obtain a quantum Kosterlitz-Thouless transition in a superfluid, we need a system with 2D space-time. To that end, consider superfluid in a cylindrical 1D channel of radius r, assuming that the wavefunction is constant in each cross-section and depends only on the coordinate x along the channel. Derive an effective 2D XY problem and find the critical point, i.e. a relation between the constants a, b, and r corresponding to the Kosterlitz-Thouless criterion for vortex pairs unbinding.

Discuss the meaning of vortices in the quantum problem. Show that they correspond to "phase slips," i.e. phase fluctuations in time such that the phase difference across a narrow region jumps by $\pm 2\pi$. What is the interpretation of the disordered and ordered state of the XY model for this system?