8.334: Statistical Mechanics II Problem Set # 1 Due: **2/12/03**

Long-range order, symmetry and soft modes

1. Superfluid dynamics

Interacting Bose gas in a superfluid state is described by the Hamiltonian

$$\mathcal{H} = \int \left(\frac{\hbar^2}{2m} |\nabla \psi|^2 + \frac{g}{2} (\bar{\psi}\psi - n)^2 \right) d^3 r \tag{1}$$

with $\psi(r)$ the Bose condensate wavefunction. Here m, g and n are particle mass, interaction, and density.

a) Find the ground state. Compare the symmetry of the Hamiltonian and the symmetry of the ground state. How many soft modes do you expect to have in this system?

b) The dynamics is described by the Gross-Pitaevsky equation[1]

$$i\hbar \partial_t \psi = \frac{\delta \mathcal{H}}{\delta \bar{\psi}} = -\frac{\hbar^2}{2m} \nabla^2 \psi + g(\bar{\psi}\psi - n)\psi \tag{2}$$

The equation for $\bar{\psi}$ is a complex conjugate of Eq.(2), $-i\hbar\partial_t \bar{\psi} = \delta\mathcal{H}/\delta\psi$. Consider a perturbed ground state $\psi(r,t) = \psi^{(0)} + \delta\psi(r,t)$ and linearize in $\delta\psi(r,t)$ the pair of equations for ψ and $\bar{\psi}$ (Eq.(2) and its conjugate). Find a plane wave solution and characterize soft modes in a superfluid. How does the dispersion relation $\omega(k)$ look at small k?

c) Treating the excitations found in part b) as Bose particles, find the entropy and specific heat of the superfluid (1) at low temperature $T \ll gn$.

2. Spin waves in a Heisenberg ferromagnet

Consider a classical spin model of a ferromagnet

$$\mathcal{H} = -\frac{1}{2} \sum_{r \neq r'} J(r - r')\, \mathbf{S}_r \cdot \mathbf{S}_{r'} \tag{3}$$

Here $\mathbf{S}_r$ is a unit vector and $J(r - r') > 0$ is spin exchange interaction. This model describes a large spin limit of the quantum Heisenberg problem.

a) Find the ground state. Compare the symmetry of the Hamiltonian (3) and the symmetry of the ground state. How many soft modes do you expect to have in this system?

b) The dynamics of each spin is precession in the magnetic field of neighboring spins,

$$\partial_t \mathbf{S}_r = \mathbf{S}_r \times \mathbf{B}_r\,, \qquad \mathbf{B}_r \equiv -\delta\mathcal{H}/\delta\,\mathbf{S}_r = \sum_{r'} J(r - r')\,\mathbf{S}_{r'} \tag{4}$$

Consider a long wavelength spin wave, $\mathbf{S}_r(t) = \mathbf{S}_r^{(0)} + \delta\,\mathbf{S}_r(t)$, linearize Eqs.(4) in $\delta\,\mathbf{S}$, and find a plane wave solution for spin waves. How does the dispersion relation $\omega(k)$ look at small k?

c) Treating spin waves as Bose particles, find the entropy and specific heat of the ferromagnet (3) at low temperature $T \ll J$.

d) Spin system in external magnetic field is described by $\mathcal{H} \to \mathcal{H} - \sum_r \mu\,\mathbf{S}_r \cdot \mathbf{B}_{ext}$ with μ the spin magnetic moment (Bohr's magneton for electron spin). Consider once more the above questions a), b) and c). Comment on the difference with the $B_{ext} = 0$ case.

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[1] This equation is also known as the nonlinear Schrödinger equation and the time-dependent Ginzburg-Landau equation.