Lecture Six

6 Vortex-Unbinding Transition in the 2D-XY Model

In this lecture we concentrate on a specific model: the XY model in two dimensions. As this model has a continuous symmetry, we know from Lecture 3 that there will only be quasi-long-range order at low temperatures. However there is still a phase transition from this state to the high-temperature state with short-range correlations, and this transition is known as the Kosterlitz-Thouless vortex-unbinding transition. We will see that a vortex is the important topological defect in the XY system, and that when free vortices proliferate due to thermal fluctuations, the system loses "spin-stiffness" and is short-range ordered. Finally we will look at the RG treatment of this phase transition, and derive a non-trivial flow diagram which gives us the critical properties near the Kosterlitz-Thouless transition. We will find this RG flow by first mapping the system of vortices to the so-called Sine-Gordon model, on which we can use a similar method of coarse-graining to that used in Lecture 5.

6.1 The XY model in two dimensions

Consider a lattice of fixed-length spins $\mathbf{S}_\mathbf{r}$ constrained to rotate only in two dimensions. With nearest-neighbour interactions the classical energy of configurations is,

$$F_{\mathrm{XY}} = -K \sum_{\langle \mathbf{r},\mathbf{r}' \rangle} \mathbf{S}_\mathbf{r} \cdot \mathbf{S}_{\mathbf{r}'} = -K \sum_{\langle \mathbf{r},\mathbf{r}' \rangle} \cos\left(\theta_\mathbf{r} - \theta_{\mathbf{r}'}\right), \tag{212}$$

where $\theta_\mathbf{r}$ is the angle of the spin relative to some fixed axis. (Note that this is the same as the discrete version of a $U(1)$ Landau theory when we are at low enough temperatures to ignore amplitude fluctuations.) It is important to state that there is no exact solution of this model in two dimensions (unlike the case for the 2D Ising model). Nevertheless, through a series of reasonable approximations, some detailed and fascinating predictions have been made.

6.1.1 Spin-wave expansion at low temperatures

At small T/K there will only be small distortions between neighbouring sites, and we can expand the cosine,

$$F_{\mathrm{XY}} = \text{const.} - \frac{K}{2} \sum_{\mathbf{r}} \sum_{\mathbf{l}} \left(\Delta_\mathbf{l} \theta_\mathbf{r}\right)^2 + \mathcal{O}(T^2/K), \tag{213}$$

where $\mathbf{l}$ label the neighbouring sites and, $\Delta_\mathbf{l}\theta_\mathbf{r} = \theta_{\mathbf{r}+\mathbf{l}} - \theta_\mathbf{r}$. This is now a quadratic theory, and therefore solvable. With the Fourier transform, $\theta_\mathbf{r} = (2\pi)^{-2} \int d^2k \; e^{i\mathbf{k}\cdot\mathbf{r}} \theta_\mathbf{k}$ we have,

$$F_{\mathrm{XY}} = \text{const.} + K \int \frac{d^2k}{(2\pi)^2} l^2 \sum_{\mathbf{l}} \left(1 - \cos \mathbf{k}\cdot\mathbf{l}\right) |\theta_\mathbf{k}|^2. \tag{214}$$

Taking the long-wavelength limit gives the elastic form,

$$F_{\mathrm{SW}} = \frac{K}{2} \int \frac{d^2k}{(2\pi)^2} \; k^2 |\theta_\mathbf{k}|^2. \tag{215}$$

Note that we have already studied this problem in Section ??. Using the elastic energy (83) we found the propagators (86) for different dimensions. In two dimensions this gave the result for the spin correlation function,

$$\langle \mathbf{S_r} \cdot \mathbf{S_{r'}} \rangle = \left(\frac{|\mathbf{r} - \mathbf{r}'|}{l} \right)^{-T/2\pi K} . \tag{216}$$

Therefore there is only quasi-long-range order at low temperatures. Note that the scale invariance of this correlation function implies that each value of T/K is a fixed point, and there is a *line* of fixed points at low temperatures.

6.1.2 High temperature expansion

We briefly review the high-temperature expansion of the XY model, to show that there is short-range order.[24] To take the high-temperature limit, we want to expand in powers of β. The XY partition function is,

$$\begin{aligned} Z_{\text{XY}} &= \int \prod_{\mathbf{r}''} d\theta_{\mathbf{r}''} e^{\beta K \sum_{\langle \mathbf{r},\mathbf{r}'\rangle} \cos(\theta_{\mathbf{r}} - \theta_{\mathbf{r}'})} \\ &= \int \prod_{\mathbf{r}''} d\theta_{\mathbf{r}''} \prod_{\langle \mathbf{r},\mathbf{r}'\rangle} e^{\beta K \cos(\theta_{\mathbf{r}} - \theta_{\mathbf{r}'})} . \end{aligned} \tag{217}$$

The function $e^{\beta K \cos x}$ is a periodic, even function of x, and can be written in a Fourier series. It is straightforward to show that,

$$e^{\beta K \cos x} = \sum_{n=0}^{\infty} I_n(\beta K) e^{inx}, \tag{218}$$

where $I_n(x)$ is a modified Bessel function with the series expansion,

$$I_n(x) = \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!} \left(\frac{x}{2} \right)^{n+2k} . \tag{219}$$

If we only keep the lowest order term in βK for each Bessel function, we find the high-temperature limit of the partition function,

$$\lim_{\beta K \to 0} Z_{\text{XY}} = \int \prod_{\mathbf{r}''} d\theta_{\mathbf{r}''} \prod_{\langle \mathbf{r},\mathbf{r}'\rangle} \left[1 + \sum_{n_{r,r'}=1}^{\infty} \frac{1}{n_{r,r'}!} \left(\frac{\beta K}{2} \right)^{n_{r,r'}} e^{in_{r,r'}(\theta_{\mathbf{r}} - \theta_{\mathbf{r}'})} \right] \tag{220}$$

Performing the integrals over $\theta_{\mathbf{r}''}$ will only get non-zero results for configurations satisfying $\nabla \cdot \mathbf{n} \equiv \sum_{\mathbf{l}} n_{r,r+l} = 0$. The partition function is then a constrained sum over possible configurations of the integers $n_{r,r'}$ defined on each bond between neighbours,

$$Z_{\text{XY}} = \text{Tr}_{\mathbf{n}} \prod_{\langle \mathbf{r},\mathbf{r}'\rangle} \frac{1}{n_{r,r'}!} \left(\frac{\beta K}{2} \right)^{n_{r,r'}} . \tag{221}$$

In the same way we can calculate the correlation function,

$$\langle \mathbf{S}_{\mathbf{r}_1} \cdot \mathbf{S}_{\mathbf{r}_2} \rangle = \left\langle e^{i(\theta_{\mathbf{r}_1} - \theta_{\mathbf{r}_2})} \right\rangle = \frac{1}{Z_{\text{XY}}} \text{Tr}'_{\mathbf{n}} \prod_{\langle \mathbf{r},\mathbf{r}'\rangle} \frac{1}{n_{r,r'}!} \left(\frac{\beta K}{2} \right)^{n_{r,r'}} . \tag{222}$$

[24] See section 4.2.1 of C. Itzykson and J.-M. Drouffe, *Statistical Field Theory,* Vol. 1 (Cambridge University Press, 1989).

where the trace Tr′ is over configurations satisfying $\nabla \cdot \mathbf{n} = \delta_{\mathbf{r},\mathbf{r}_1} - \delta_{\mathbf{r},\mathbf{r}_2}$. The main contribution (lowest order in β) comes from the links on the shortest path between the two points $\mathbf{r}_1$ and $\mathbf{r}_2$ having $n = 1$ with all other links zero. This gives,

$$\langle \mathbf{S}_{\mathbf{r}_1} \cdot \mathbf{S}_{\mathbf{r}_2} \rangle = \prod_{\langle \mathbf{r},\mathbf{r}' \rangle} \left(\frac{\beta K}{2} \right) = \left(\frac{\beta K}{2} \right)^{|\mathbf{r}_1 - \mathbf{r}_2|/l} = e^{-|\mathbf{r}_1 - \mathbf{r}_2|/\xi}. \tag{223}$$

Therefore we have an exponential decay in the correlation function, with $\xi = l/\ln(2T/K)$.

The different functional forms of the correlation functions at high and low temperatures implies that there must be a phase transition between the two behaviours, as was first pointed out by Berezinskii,[25] who also showed that the low temperature phase has an infinite magnetic susceptibility and a rigidity to long-wavelength spin distortions (spin-stiffness).

6.1.3 Separation of longitudinal and transverse fluctuations (spin-waves and vortices)

Consider the change in spin-angle as we follow a path C,

$$\Delta\theta = \sum_{\langle r,r' \rangle}^{C} (\theta_{\mathbf{r}} - \theta_{\mathbf{r}'}) \approx \int_C d\mathbf{r} \cdot \nabla\theta_{\mathbf{r}}. \tag{224}$$

If C is a closed path, then the change in angle must be a multiple of 2π,

$$\oint_C d\mathbf{r} \cdot \nabla\theta_{\mathbf{r}} = 2n_c\pi. \tag{225}$$

Note that the fact that this result must be unchanged for small deformations in the path C, together with Stokes' theorem tells us that (except for isolated points),

$$\nabla \times \nabla\theta_{\mathbf{r}} = 0. \tag{226}$$

Within this continuum limit, this means that a non-zero result for (225) can only be obtained from singular points, which define the "core" of a topological defect, the vortex. Therefore we should generalize (226) to include singularities of strength n_i located at points $\mathbf{r}_i$, so that

$$\nabla \times \nabla\theta_{\mathbf{r}} = 2\pi\hat{\mathbf{z}} \sum_i n_i \delta^2(\mathbf{r} - \mathbf{r}_i) \equiv 2\pi\mathbf{n}(\mathbf{r}). \tag{227}$$

This defines the vortex density $\mathbf{n}(\mathbf{r})$.

In the continuum limit, the energy of the system only depends on the spatially varying gradient of phase, $\nabla\theta_{\mathbf{r}}$. By a quantum-mechanical analogy, this gradient looks a bit like a velocity, so we write $\mathbf{v}_{\mathbf{r}} \equiv \nabla\theta_{\mathbf{r}}$. It will be useful to write this velocity as a sum of "longitudinal" and "transverse" parts, $\mathbf{v}_{\mathbf{r}} = \mathbf{v}_{\mathbf{r}}^l + \mathbf{v}_{\mathbf{r}}^t$ defined by,

$$\begin{aligned} \nabla \times \mathbf{v}_{\mathbf{r}}^l &= 0, \\ \nabla \cdot \mathbf{v}_{\mathbf{r}}^t &= 0. \end{aligned} \tag{228}$$

Note that this means that there are no vortex contributions to the longitudinal part, while the transverse part is entirely determined by the positions of the vortex cores,

$$\mathbf{v}_{\mathbf{r}}^t = 2\pi \int d^d r' G_L(\mathbf{r} - \mathbf{r}') \nabla \times \mathbf{n}(\mathbf{r}'), \tag{229}$$

[25] V.L. Berezinskii, Sov. Phys. JETP **32**, 493 (1971); Sov. Phys. JETP **34**, 610 (1971).

where $G_L(\mathbf{r}-\mathbf{r}')$ is the Green's function for the Laplacian operator $-\nabla^2$.

In the low T regime where we have expanded the cosine, we can then separate the contributions of transverse and longitudinal parts to the energy $F_{\rm XY} = F^l_{\rm XY} + F^t_{\rm XY}$, We then get a "spin-wave" contribution,

$$F^l_{\rm XY} = \int d^2r \frac{K}{2}|\mathbf{v}^l_{\mathbf{r}}|^2 = \int d^2r \frac{K}{2}|\nabla\theta^l_{\mathbf{r}}|^2, \tag{230}$$

and a "vortex" contribution,

$$F^t_{\rm XY} = \frac{(2\pi)^2K}{2}\int d^2r \int d^2r' G_L(\mathbf{r}-\mathbf{r}')\mathbf{n}(\mathbf{r})\cdot\mathbf{n}(\mathbf{r}'). \tag{231}$$

This contribution is sometimes called the "Coulomb gas" because of the form of the pairwise interactions.

Our general approach then will be to treat separately the spin-wave part from the Coulomb gas (this is only a good approximation at low temperatures). Therefore the XY model reduces to the problem of solving for the Coulomb gas. We will repose the Coulomb gas in a discrete form, with a general configuration given by the number $n_{\mathbf{r}}$ of charges at the site $\mathbf{r}$, so that the partition function is

$$Z_{\rm CG} = \prod_{\mathbf{r}} \sum_{n_{\mathbf{r}}=-\infty}^{\infty} e^{-\beta F_{\rm CG}[n_{\mathbf{r}}]}, \tag{232}$$

with the configuration energy,

$$F_{\rm CG}[n_{\mathbf{r}}] = \frac{(2\pi)^2K}{2}\sum_{\mathbf{r}}\sum_{\mathbf{r}'} n_{\mathbf{r}}n_{\mathbf{r}'}G_L(\mathbf{r}-\mathbf{r}') + \sum_{\mathbf{r}} \varepsilon_c n^2_{\mathbf{r}} \tag{233}$$

where we have included the core correction to the energy of each vortex ε_c, due to details in the lattice not captured by the continuum approximation.

6.1.4 Spin-stiffness and vortex density

The spin-stiffness of the system can be defined by the response to a force that tries to impose a relative twist of the spins at the boundary of the system. We can define a force $\mathbf{f}$ by including a linear energy gain for the spin twist, $-f\Delta\theta$, with a relative twist,

$$\Delta\theta \equiv \langle(\theta_L - \theta_0)\rangle = \int_0^L dx \langle\partial_x\theta_x\rangle. \tag{234}$$

The spin stiffness $K_{\rm eff}$ is then defined by,

$$\Delta\theta = \frac{f}{K_{\rm eff}}. \tag{235}$$

Also note that the energy change in the presence of the force will be

$$\Delta F = \frac{f^2}{2K_{\rm eff}}. \tag{236}$$

For an elastic (or harmonic) theory such as (215) for the spin-wave part, the response is easily derived to find a temperature independent spin stiffness K. However, the XY model also has higher-order (anharmonic) terms, which for our purposes are

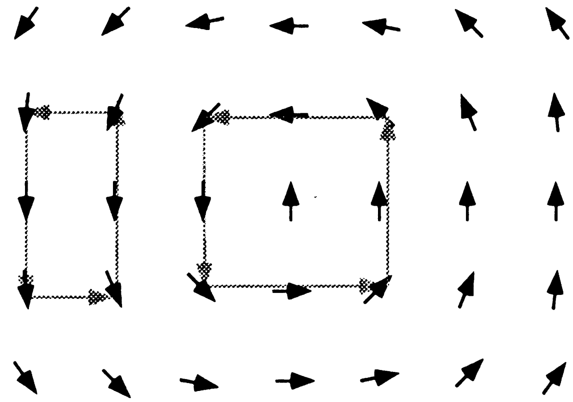

Figure 26: An example spin-configuration in a vortex in the XY model. Notice the the change in angle by 2π for a path that encloses the vortex core.

captured in the transverse (vortex) modes (233). By treating longitudinal and transverse modes separately, it is straightforward to derive the free energy change in the presence of a twisting force,

$$
\begin{aligned}
F(\mathbf{f}) &= F(0) + \frac{f^2}{2K} + \frac{f^2}{2TL^2}\int d^2r_1 \int d^2r_1 \langle \mathbf{v}^t_{\mathbf{r}_1} \cdot \mathbf{v}^t_{\mathbf{r}_2}\rangle + \mathcal{O}(f^4) & (237)\\
&= F(0) + \frac{f^2}{2K} + \frac{2\pi^2 f^2}{T}\int d^2r\; r^2 \langle \mathbf{n}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{0})\rangle + \mathcal{O}(f^4), & (238)
\end{aligned}
$$

where the last line uses the vortex density $\mathbf{n}(\mathbf{r}) = (2\pi)^{-1}\nabla \times \mathbf{v}^t_{\mathbf{r}}$. Therefore the effective stiffness must be given by,

$$
\frac{1}{K_{\text{eff}}} = \frac{1}{K} + \frac{4\pi^2}{T}\int d^2r\; r^2\langle \mathbf{n}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{0})\rangle. \tag{239}
$$

It turns out that the integral of the vortex-density correlation function is negative (due to vortex-dipole correlations) so that the net effect of a finite density of vortex–anti-vortex pairs is to reduce the effective spin-stiffness. One can also show that the same K_{eff} appears in the exponent for the correlation function when we include the transverse modes.

6.1.5 Structure and energy of a vortex

We now look at the vortex solution in more detail. For example, consider a vortex centred at the origin. The phase-field will be the solution to,

$$
\nabla \times \nabla\theta_{\mathbf{r}} = 2\pi\delta^2(r), \tag{240}
$$

which has the solution

$$
\theta^1_{\mathbf{r}} = \tan^{-1}(y/x), \tag{241}
$$

with a "velocity" of,

$$
\mathbf{v}^1_{\mathbf{r}} = \nabla\theta^1_{\mathbf{r}} = \frac{\hat{\mathbf{z}} \times \hat{\mathbf{r}}}{r}. \tag{242}
$$

An example is shown in Fig. 26. The energy of such a configuration is given by,

$$
F^{\text{v}}_{\text{XY}} = \frac{K}{2}\int d^2r|\nabla\theta^1_{\mathbf{r}}|^2 = \frac{K}{2}\int d^2r\frac{1}{r^2} = \pi K \ln(L/l), \tag{243}
$$

where we must include the cut-offs in the limits of the integral as the system size, L, and the lattice length l. Notice that a vortex costs an infinite energy in an infinitely big system.

For more than one vortex, we just add the solutions to (240) for different source terms. This leads to the pairwise dependence of the energy given above in (233). As an example consider a pair of oppositely charged vortices each with a strength n. Writing $\mathbf{v_r} = \mathbf{v}^1_{\mathbf{r}-\mathbf{r}_1} - \mathbf{v}^1_{\mathbf{r}-\mathbf{r}_2}$ and using (242) gives the energy of the pair,

$$\begin{aligned} F^{\text{pair}}_{\text{XY}}(\mathbf{r}-\mathbf{r_2}) &= \frac{n^2 K}{2}\int d^2 r \left| \frac{\mathbf{r}-\mathbf{r_1}}{|\mathbf{r}-\mathbf{r_1}|^2} - \frac{\mathbf{r}-\mathbf{r_2}}{|\mathbf{r}-\mathbf{r_2}|^2}\right|^2 \\ &= \frac{n^2 K}{2}\int \frac{d^2 k}{(2\pi)^2}\frac{|e^{i\mathbf{k}\cdot\mathbf{r}_1} - e^{i\mathbf{k}\cdot\mathbf{r}_2}|^2}{k^2} \\ &= 2\pi n^2 K \ln(|\mathbf{r}-\mathbf{r_2}|/l). \end{aligned} \tag{244}$$

So we see that oppositely charged vortices attract, and that a pair only costs a finite energy (the diverging energy from currents at far distances has cancelled). This result agrees with the general form of (233), when we take the continuum limit $G_L(r) = (2\pi)^{-1}\ln(r/L)$.

6.1.6 Single-vortex derivation of Kosterlitz-Thouless transition

Naively, we might suggest that because a vortex costs an infinite energy to put into an infinite system, there will never be a finite density of these objects from thermal fluctuations. However, this is seen not to be the case when we consider the *free energy* change to add a vortex.[26] Assuming that the entropy of a vortex is just the logarithm of the number of places we can put it, we get

$$S_v = k_B \ln[(L/l)^2], \tag{245}$$

and

$$F_v = (\pi K - 2k_B T)\ln(L/l). \tag{246}$$

Therefore we can reduce the free energy by adding a vortex to a vortex-free system as long as the temperature is above,

$$T_{\text{KT}} = \frac{\pi}{2}K. \tag{247}$$

Although the real system becomes much more complicated when we need to consider more than one vortex, this argument certainly shows that there is an instability towards vortex proliferation at this "Kosterlitz-Thouless" temperature.

The fuller picture that has emerged of this Kosterlitz-Thouless transition is one where at low temperatures there is a small but finite density of bound vortex-pairs, which may renormalize the spin-stiffness, but do not destroy the quasi-long-range order. Above the transition however, each vortex is essentially "free", so that its wanderings lead to violent phase fluctuations that kill the algebraic order and reduce it to a short-range order with correlation length of order the average spacing of the vortices.

A detailed derivation of the transition was given by Kosterlitz,[27] who used a real-space RG scaling in the low temperature phase to integrate out small length scales.

[26] This argument is due to J.M. Kosterlitz and D.J. Thouless, J. Phys. C **6**, 1181 (1973).

[27] J.M. Kosterlitz, J. Phys. C **7**, 1046 (1974).

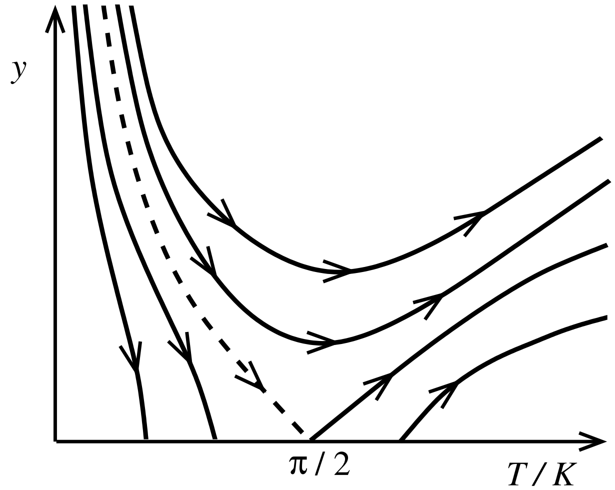

Figure 27: Schematic RG flow diagram for the XY model, for the parameters $y = e^{-\varepsilon_c/T}$ and T/K. Note that $y = 0$ is a line of fixed points, which are stable for $T < \pi K/2$. The dashed line marks the actual phase boundary between a low-temperature phase with no vortices and a high-temperature phase with free vortices.

The effect of the coarse-graining is to renormalize the core energy of a vortex. In the low temperature phase, this core-energy flows to infinity, and the effective coarse-grained system has no vortices. Above the transition, however, the core energy flows to zero, which implies a rush of vortices and a breakdown of the ordered phase. Rather than following the Kosterlitz procedure, we will derive the same RG flow diagram in the next section by mapping the vortex problem to a different model, where a momentum-space RG method can be used. For now we just look at the resulting schematic RG flow diagram for parameters T/K versus $y \equiv e^{-\varepsilon_c/T}$ in Fig. 27. Note how the line $y = 0$, corresponding to no vortices, is a line of fixed points. Below the temperature $T = \pi K/2$ these fixed points are stable to small values of y, whereas above this temperature the fixed points are unstable. For non-zero y, the flow is always to larger temperatures. Note the critical line that flows to the point $(y = 0, T = \pi K/2)$. This marks the actual phase transition: as the real system parameters cross through this line in the (y, T) plane, the behaviour swaps from the low-temperature phase with no vortices to the high temperature phase with vortices.

6.2 The Sine-Gordon model in two dimensions

Starting from the partition function for the Coulomb gas of (232), we will find that in a certain limit it is equivalent to the partition function for the so-called Sine-Gordon model, defined by the energy functional,

$$\beta F_{\mathrm{SG}}[\phi] = \int d^2r \left[\frac{g}{2}|\partial_{\mathbf{r}}\phi|^2 - J\cos\phi_{\mathbf{r}}\right]. \tag{248}$$

6.2.1 Duality transformation from Coulomb gas to Sine-Gordon

We first rewrite the interaction part of the Coulomb gas energy (233) in Fourier space,

$$e^{-\frac{1}{2}(2\pi)^2\beta K\sum_{\mathbf{r},\mathbf{r}'} n_{\mathbf{r}} n_{\mathbf{r}'} G_L(\mathbf{r}-\mathbf{r}')} = e^{-\frac{1}{2}(2\pi)^2\beta K\frac{1}{L^2 l^4}\sum_{\mathbf{k}}|n_{\mathbf{k}}|^2 G_L(\mathbf{k})}, \tag{249}$$

with $G_L(\mathbf{k}) = l^2/\sum_{\mathbf{l}}(1-\cos\mathbf{k}\cdot\mathbf{l})$. Then we introduce an auxiliary phase,

$$e^{-\frac{1}{2}(2\pi)^2\beta K\frac{1}{L^2l^4}|n_{\mathbf{k}}|^2G_L(\mathbf{k})} = A_{\mathbf{k}}\int_{-\infty}^{\infty}d\phi e^{\frac{1}{L^2}\left(-\frac{1}{2}\frac{|\phi|^2}{(2\pi)^2\beta KG_L(\mathbf{k})}+i\frac{1}{l^2}\phi n_{\mathbf{k}}\right)}. \tag{250}$$

Doing this for each $\mathbf{k}$ vector, and then transforming back to real space gives the identity for the partition function of (232),

$$Z_{\rm CG} = A'\left(\prod_{\mathbf{r}'}\sum_{n_{\mathbf{r}'}=-\infty}^{\infty}\right)\left(\int_{-\infty}^{\infty}\prod_{\mathbf{r}''}d\phi_{\mathbf{r}''}\right)e^{-\frac{1}{2}\frac{1}{(2\pi)^2\beta K}\sum_{\mathbf{r},\mathbf{l}}(\Delta_{\mathbf{l}}\phi_{\mathbf{r}})^2+\sum_{\mathbf{r}}\phi_{\mathbf{r}}n_{\mathbf{r}}-\beta\varepsilon_c\sum_{\mathbf{r}}n_{\mathbf{r}}^2}. \tag{251}$$

The Kosterlitz RG results were obtained in the limit of very large core energy. Therefore we will consider the same limit, the effect of which is first to restrict the allowed values of $n_{\mathbf{r}}$ to be only $[0, \pm 1]$. (the density of sites with $|n_{\mathbf{r}}| = 2$ will be a factor of $\exp(-\beta\varepsilon_c)$ smaller). This allows us to complete the trace over $n_{\mathbf{r}}$ in (251),

$$\sum_{n=-1}^{1} e^{i\phi n-\beta\varepsilon_c n^2} = 1 + 2e^{-\beta\varepsilon_c}\cos\phi. \tag{252}$$

We will label the exponentially small factor as $e^{-\beta\varepsilon_c} \equiv y$ (this is known as the fugacity), and after reexponentiating we have,

$$\sum_n e^{i\phi n-\beta\varepsilon_c n^2} = e^{2y\cos\phi} + \mathcal{O}(y^2). \tag{253}$$

Including this result in (251) gives the partition function as a trace over the auxiliary phase only,

$$Z_{\rm CG} = A'\int_{-\infty}^{\infty}\prod_{\mathbf{r}''}d\phi_{\mathbf{r}''}e^{-\frac{1}{2}\frac{1}{(2\pi)^2\beta K}\sum_{\mathbf{r},\mathbf{l}}(\Delta_{\mathbf{l}}\phi_{\mathbf{r}})^2+\sum_{\mathbf{r}}2y\cos\phi_{\mathbf{r}}}. \tag{254}$$

This is just the form of the partition function for the discrete version of the Sine-Gordon model (248). In fact, we can write the relationship between the Coulomb gas defined by $Z_{\rm CG}(\beta K, \beta\varepsilon_c)$ and the Sine-Gordon model $Z_{\rm SG}(g, J)$ as,

$$Z_{\rm CG}(\beta K, \beta\varepsilon_c) = A' Z_{\rm SG}\left(\frac{1}{(2\pi)^2\beta K}, \frac{2}{l^2}e^{-\beta\varepsilon_c}\right) \tag{255}$$

This is known as a duality transformation, where every point in the low-temperature phase of the Coulomb gas maps to a point in the high-temperature phase in the Sine-Gordon model, and vice-versa.

Our knowledge that there is a phase-transition in the 2D-XY model tells us that there must be a phase transition in the 2D-Sine-Gordon model. However, in the second case the high-temperature phase must be the one with algebraic correlations and quasi-long-range order. This is clearly true for the model (248) if J can be ignored. So at high temperatures, thermal fluctuations in the Sine-Gordon model drive the effective J to zero (just as the fugacity $e^{-\beta\varepsilon_c}$ flows to zero in the low-temperature Coulomb gas). Similarly, at low temperatures we expect a finite correlation length in the Sine-Gordon model, which in this case is in an ordered phase, with J remaining "relevant" (this corresponds to a relevant fugacity, or small effective vortex-core energy in the high temperature phase of the Coulomb gas). We will now substitute this hand-waving discussion for an RG calculation of the Sine-Gordon model.

6.2.2 Momentum-space RG for Sine-Gordon model

First we note that the continuum action (248) should be defined only down to a small-length cut-off $l = 2\pi\Lambda^{-1}$. We will coarse-grain in the same way as in Lecture 5, by separating a shell of small-wavelength modes out of the Fourier transform, i.e. $\phi_{\mathbf{r}} = \phi^<_{\mathbf{r}} + \phi^>_{\mathbf{r}}$, with,

$$\begin{aligned} \phi^<_{\mathbf{r}} &= \int_{0<q<\Lambda/\lambda} \frac{d^2q}{(2\pi)^2} e^{i\mathbf{q}\cdot\mathbf{r}} \phi_{\mathbf{q}}, \\ \phi^>_{\mathbf{r}} &= \int_{\Lambda/\lambda<q<\Lambda} \frac{d^2q}{(2\pi)^2} e^{i\mathbf{q}\cdot\mathbf{r}} \phi_{\mathbf{q}}. \end{aligned} \tag{256}$$

Then we can write the Sine-Gordon action as a functional of the fast and slow modes,

$$\beta F_{\rm SG}[\phi^<, \phi^>] = \int d^2r \left[\frac{g}{2} |\partial_{\mathbf{r}}\phi^<|^2 + \frac{g}{2} |\partial_{\mathbf{r}}\phi^>|^2 - J\cos(\phi^<_{\mathbf{r}} + \phi^>_{\mathbf{r}}) \right]. \tag{257}$$

To coarse-grain, we take a partial trace over the fast modes to leave an effective action of the slow modes,

$$\begin{aligned} e^{-\beta F'_{\rm SG}[\phi^<]} &= \int \prod_{\Lambda/\lambda<q<\Lambda} d\phi^>_{\mathbf{q}}\; e^{-\beta F_{\rm SG}[\phi^<,\phi^>]} \\ &= e^{-\int d^2r \frac{g}{2}|\partial_{\mathbf{r}}\phi^<|^2} \int \prod_{\Lambda/\lambda<q<\Lambda} d\phi^>_{\mathbf{q}}\; e^{-\int d^2r \frac{g}{2}|\partial_{\mathbf{r}}\phi^>|^2 - J\cos(\phi^<_{\mathbf{r}}+\phi^>_{\mathbf{r}})} \\ &= e^{-\int d^2r \frac{g}{2}|\partial_{\mathbf{r}}\phi^<|^2} Z_> \left\langle e^{\int d^2r J\cos(\phi^<_{\mathbf{r}}+\phi^>_{\mathbf{r}})} \right\rangle_>, \end{aligned} \tag{258}$$

where $Z_>$ and $\langle\ldots\rangle_>$ are averages over the fast modes with respect to the Gaussian weight $\exp(-\int d^2r\, \frac{g}{2}|\partial_{\mathbf{r}}\phi^>|^2)$. We have written it in this form so as to be able to attempt the Gaussian averages.

6.2.3 Important averages over the fast modes

As we are interested in the small J limit, we we will expand the average in (258) to get,

$$\begin{aligned} \left\langle e^{\int d^2r J\cos(\phi^<_{\mathbf{r}}+\phi^>_{\mathbf{r}})} \right\rangle_> &= 1 + J\int d^2r\, \langle\cos(\phi^<_{\mathbf{r}} + \phi^>_{\mathbf{r}})\rangle_> \\ &\quad + \frac{J^2}{2}\int d^2r_1 \int d^2r_2\, \left\langle \cos(\phi^<_{\mathbf{r}_1} + \phi^>_{\mathbf{r}_1})\cos(\phi^<_{\mathbf{r}_2} + \phi^>_{\mathbf{r}_2}) \right\rangle_> + \mathcal{O}(J^3) \\ &= e^{J\int d^2r \langle\cos(\phi^<_{\mathbf{r}}+\phi^>_{\mathbf{r}})\rangle_> + \frac{J^2}{2}\int d^2r_1 \int d^2r_2 \left[\left\langle \cos(\phi^<_{\mathbf{r}_1}+\phi^>_{\mathbf{r}_1})\cos(\phi^<_{\mathbf{r}_2}+\phi^>_{\mathbf{r}_2}) \right\rangle_> - \left\langle\cos(\phi^<_{\mathbf{r}_1}+\phi^>_{\mathbf{r}_1})\right\rangle_> \left\langle\cos(\phi^<_{\mathbf{r}_2}+\phi^>_{\mathbf{r}_2})\right\rangle_> \right]}. \end{aligned} \tag{259}$$

We therefore need the cosine average,

$$\begin{aligned} \langle\cos(\phi^<_{\mathbf{r}} + \phi^>_{\mathbf{r}})\rangle_> &= \frac{1}{2} e^{i\phi^<_{\mathbf{r}}} \langle e^{i\phi^>_{\mathbf{r}}}\rangle_> + \frac{1}{2} e^{-i\phi^<_{\mathbf{r}}} \langle e^{-i\phi^>_{\mathbf{r}}}\rangle_> \\ &= e^{-\frac{1}{2}g_>(0)} \cos\phi^<_{\mathbf{r}}, \end{aligned} \tag{260}$$

where we defined the "fast" propagator,

$$g_>(\mathbf{r}) = \int_{0<q<\Lambda/\lambda} \frac{d^2q}{(2\pi)^2} \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{g\,q^2}. \tag{261}$$

We also need the double cosine average,

$$\begin{aligned}\left\langle\cos(\phi^<_{\mathbf{r}_1}+\phi^>_{\mathbf{r}_1})\cos(\phi^<_{\mathbf{r}_2}+\phi^>_{\mathbf{r}_2})\right\rangle_> &= \frac{1}{4}\left\{e^{i(\phi^<_{\mathbf{r}_1}+\phi^<_{\mathbf{r}_2})}\langle e^{i(\phi^>_{\mathbf{r}_1}+\phi^>_{\mathbf{r}_2})}\rangle_> + e^{-i(\phi^<_{\mathbf{r}_1}+\phi^<_{\mathbf{r}_2})}\langle e^{-i(\phi^>_{\mathbf{r}_1}+\phi^>_{\mathbf{r}_2})}\rangle_> + \right.\\ &\left. + e^{i(\phi^<_{\mathbf{r}_1}-\phi^<_{\mathbf{r}_2})}\langle e^{i(\phi^>_{\mathbf{r}_1}-\phi^>_{\mathbf{r}_2})}\rangle_> + e^{-i(\phi^<_{\mathbf{r}_1}-\phi^<_{\mathbf{r}_2})}\langle e^{-i(\phi^>_{\mathbf{r}_1}-\phi^>_{\mathbf{r}_2})}\rangle_>\right\}\\ &= \frac{1}{2}\left[e^{-[g_>(0)+g_>(\mathbf{r}_1-\mathbf{r}_2)]}\cos(\phi^<_{\mathbf{r}_1}+\phi^<_{\mathbf{r}_2})\right.\\ &\qquad \left.+e^{-[g_>(0)-g_>(\mathbf{r}_1-\mathbf{r}_2)]}\cos(\phi^<_{\mathbf{r}_1}-\phi^<_{\mathbf{r}_2})\right]. \qquad (262)\end{aligned}$$

Including these results in the expansion (259) gives,

$$\begin{aligned}\ln\left[\left\langle e^{\int d^2r J\cos(\phi^<_{\mathbf{r}}+\phi^>_{\mathbf{r}})}\right\rangle_>\right] &= J\int d^2r\, e^{-\frac{1}{2}g_>(0)}\cos\phi^<_{\mathbf{r}}\\ &+\frac{J^2}{4}\int d^2r_1\int d^2r_2\left\{e^{-g_>(0)}\left[e^{-g_>(\mathbf{r}_1-\mathbf{r}_2)}-1\right]\cos(\phi^<_{\mathbf{r}_1}+\phi^<_{\mathbf{r}_2})\right.\\ &\left.+e^{-g_>(0)}\left[e^{g_>(\mathbf{r}_1-\mathbf{r}_2)}-1\right]\cos(\phi^<_{\mathbf{r}_1}-\phi^<_{\mathbf{r}_2})\right\}. \qquad (263)\end{aligned}$$

At first sight, the double integral that appears to second order in J does not look like it can lead to a renormalized action of the form of (248). However, we note that the function $g_>(r)$ should become very small as r gets larger than the new cut-off scale, as only Fourier modes with $q>\Lambda/\lambda$ have any weight in the definition (261). Therefore the combination $\left[e^{\pm g_>(\mathbf{r}_1-\mathbf{r}_2)}-1\right]$ is small everywhere but $|\mathbf{r}_1-\mathbf{r}_2|\sim\lambda/\Lambda$, and we can expand it in the difference of $\mathbf{r}_1$ and $\mathbf{r}_2$. We do this using the transformations

$$\begin{aligned}\mathbf{u} &= \frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)\\ \mathbf{v} &= (\mathbf{r}_1-\mathbf{r}_2). \qquad (264)\end{aligned}$$

Then for small v we write,

$$\begin{aligned}\phi^<_{\mathbf{r}_1}+\phi^<_{\mathbf{r}_2} &\approx 2\phi^<_{\mathbf{u}}\\ \phi^<_{\mathbf{r}_1}-\phi^<_{\mathbf{r}_2} &\approx \mathbf{v}\cdot\partial_{\mathbf{u}}\phi^<_{\mathbf{u}}. \qquad (265)\end{aligned}$$

We can then approximate (263) as,

$$\begin{aligned}\ln\left[\left\langle e^{\int d^2r J\cos(\phi^<_{\mathbf{r}}+\phi^>_{\mathbf{r}})}\right\rangle_>\right] &= J\int d^2r\, e^{-\frac{1}{2}g_>(0)}\cos\phi^<_{\mathbf{r}}\\ &+\frac{J^2}{2}\int d^2r\left\{I_1\cos(2\phi^<_{\mathbf{r}})+I_2|\partial_{\mathbf{r}}\phi^<_{\mathbf{r}}|^2+\text{const.}\right\}, (266)\end{aligned}$$

with

$$\begin{aligned}I_1 &= \frac{1}{2}e^{-g_>(0)}\int d^2v\left[e^{-g_>(\mathbf{v})}-1\right],\\ I_2 &= -\frac{1}{4}e^{-g_>(0)}\int d^2v\; v^2\left[e^{g_>(\mathbf{v})}-1\right]. \qquad (267)\end{aligned}$$

Now, we will only consider the lowest-order corrections in J to the two terms appearing in the action (248). Therefore we ignore for now the term in $\cos(2\phi^<_{\mathbf{r}})$, and get the result,

$$e^{-\beta F'_{\rm SG}[\phi^<]} = X e^{-\int d^2r\left[\frac{1}{2}(g+J^2I_2)|\partial_{\mathbf{r}}\phi^<_{\mathbf{r}}|^2 - Je^{-\frac{1}{2}g_>(0)}\cos\phi^<_{\mathbf{r}}\right]}. \qquad (268)$$

6.2.4 RG equations for Sine-Gordon model

Eq. (268) tells us the effect on J and g after coarse-graining. Of course for the RG procedure, we should also rescale to get a coarse-grained system with the original cut-off. After doing this we easily find the following RG equations for a given coarse-graining of λ,

$$
\begin{aligned}
J' &= \lambda^2 J e^{-\frac{1}{2} g_>(0)}, \\
g' &= g + J^2 I_2(\lambda, g).
\end{aligned}
\tag{269}
$$

with I_2 defined in (267).

We can make more sense of the dependence of I_2 if we take the infinitesimal limit of an RG procedure, i.e., we only take out the fastest modes in an infinitesimal slice of width $d\Lambda$, so that

$$
\begin{aligned}
g_>(r) &= \int_{\Lambda - d\Lambda < q < \Lambda} \frac{d^2 q}{(2\pi)^2} \frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{g\, q^2}, \\
&= \frac{1}{2\pi g} J_0(\Lambda r) \frac{d\Lambda}{\Lambda},
\end{aligned}
\tag{270}
$$

with $J_0(x)$ the zeroth-order Bessel function. Substituting this in (267) gives,

$$
I_2 = \frac{1}{g} \frac{d\Lambda}{\Lambda^5} \alpha_2,
\tag{271}
$$

with the dimensionless constant $\alpha_2 = (8\pi)^{-1} \int d^2x\; x^2 J_0(x)$. We then get the infinitesimal form of the flow equations,

$$
\begin{aligned}
J' &= J \left(1 + 2\frac{d\Lambda}{\Lambda}\right) \left(1 - \frac{1}{4\pi g} \frac{d\Lambda}{\Lambda}\right), \\
g' &= g \left(1 + \frac{J^2 \alpha_2}{g^2} \frac{d\Lambda}{\Lambda^5}\right).
\end{aligned}
\tag{272}
$$

Writing $J' = J + dJ$ and $g' = g + dg$ turns this into,

$$
\begin{aligned}
\frac{dJ}{J} &= \left(2 - \frac{1}{4\pi g}\right) \frac{d\Lambda}{\Lambda}, \\
g dg &= J^2 \alpha_2 \frac{d\Lambda}{\Lambda^5}.
\end{aligned}
\tag{273}
$$

Physically we see that the coarse-graining process leads to an effective system with either larger or smaller J depending on the sign of $(g - 8\pi)$. For small enough g, the flow is to smaller J, and successive RG transformations take the system to a $J = 0$ fixed point, corresponding to a system with logarithmic roughness in the correlations in $\phi(\mathbf{r})$. For large enough g, the flow is to larger J, and the flow is to a bulk fixed point with zero correlation length. Also note that the flows always increase the value of g, i.e., the effective phase stiffness increases with the coarse-graining.

6.2.5 Kosterlitz flow diagram

We now make a few substitutions to cast the flow equations in a simpler form, as well as to make them look more like the form that Kosterlitz first found. We substitute

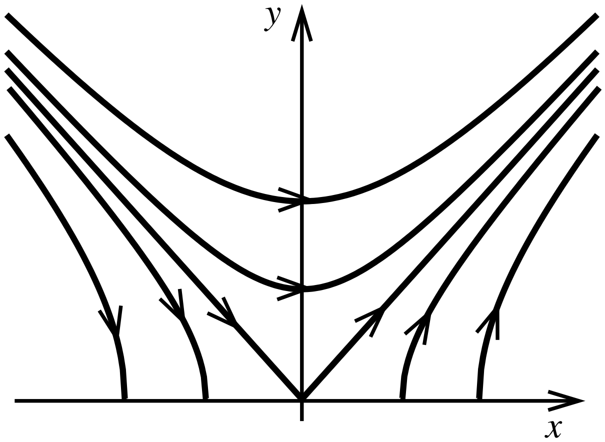

Figure 28: Calculated RG flow for small x and y satisfying the equation $x^2 - y^2 =$ const., where the constant depends on the initial parameters before renormalization. If the constant is negative, the flow is eventually to large values of y. The critical line is when the constant is zero. For a positive constant, and with $x < 0$, the flow is to $y = 0$.

the change in q-space cut-off $d\Lambda$ for the change in real-space cut-off dl. Then we get,

$$\begin{aligned} \frac{dJ}{J} &= \left(2 - \frac{1}{4\pi g}\right)\frac{dl}{l}, \\ gdg &= J^2\alpha_2 l^3 dl. \end{aligned} \tag{274}$$

Now we write $x = (1/4\pi g) - 2$, to give

$$\begin{aligned} \frac{dJ}{J} &= -x\frac{dl}{l}, \\ \frac{dx}{(x+2)^3} &= -(4\pi)^2 J^2\alpha_2 l^3 dl. \end{aligned} \tag{275}$$

Finally we substitute $y = 4\pi\sqrt{8\alpha_2}l^2 J$, so that

$$\begin{aligned} \frac{dy}{y} &= -x\frac{dl}{l}, \\ \frac{dx}{(x+2)^3} &= -\frac{1}{8}y^2\frac{dl}{l}. \end{aligned} \tag{276}$$

The two important things to notice about this set of flow equations is that $y = 0$ is a fixed point for all x (i.e. a *line* of fixed points), and that $x = 0$ is also special in that it makes $dy = 0$, (although not dx). For this reason we zoom in on the vicinity of $(x, y) = (0, 0)$, where we can solve the equations. Replacing $(x + 2)$ with 2, allows us to write,

$$-xy^2\frac{dl}{l} = xdx = ydy, \tag{277}$$

so that solutions must be of the form

$$x^2 = y^2 + \text{const.} \tag{278}$$

In other words, the flow near the origin follows the curve of a hyperbola. The particular constant depends on the initial conditions, and its sign determines whether the system flows to $y = 0$, or to infinite y. To demonstrate this, the flow lines are drawn in Fig. 28.

Therefore the phase transition between these two behaviours occurs when the constant is zero, or when the initial conditions,

$$\begin{aligned} x_0 &= \frac{1}{4\pi g} - 2, \\ y_0 &= 4\pi\sqrt{8\alpha_2} l^2 J \end{aligned} \tag{279}$$

satisfy $x_0^2 = y_0^2$. The critical value of g for the phase transition is therefore,

$$g_c = 1/8\pi - \sqrt{\alpha_2/2} l^2 J + \mathcal{O}(J^2). \tag{280}$$

Notice the strange result that in the limit $J \to 0$, the phase transition only depends on the value of g. This means that for $g < 1/8\pi$, J is always a relevant pertubation, as is seen in the RG flow we have derived.

6.3 Properties of Kosterlitz-Thouless transition from RG

We should now think about what the flow diagram we have derived for the Sine-Gordon model implies for the XY model. Remember that we have shown the partition functions to be the same if we take $g = 1/(2\pi)^2 \beta K$ and $J = 2l^{-2} e^{-\beta\varepsilon_c}$. Therefore we can write the dimensionless flow variables

$$\begin{aligned} x &= \pi\beta K - 2, \\ y &= b e^{-\beta\varepsilon_c}, \end{aligned} \tag{281}$$

with $b = 8\pi\sqrt{8\alpha_2}$. The flow diagram of Fig. 28 is then seen to represent either flows to zero fugacity at low temperature or to large fugacity at high temperature as in Fig. 27. The critical value of temperature where the two different behaviours are separated is given by, using (280),

$$T_{\mathrm{KT}} = \frac{\pi K}{2}\left[1 - \frac{b}{2} e^{-\varepsilon_c/T_{\mathrm{KT}}}\right]. \tag{282}$$

This is a self-consistent relation for T_{KT}, that needs to be solved numerically. Notice how, in the limit of very large core energy, this result converges on the simple result from the free energy consideration of a single vortex in Section 6.1.6.

In fact, a much more complete picture emerges, if we write the effective spin-stiffness as given in (239). It turns out that evaluating the vortex-density correlation function leads to a correction of exactly the same form as in (282), so that the simple Kosterlitz-Thouless result is recovered, only with the *effective* spin-stiffness in the formula,

$$T_{\mathrm{KT}} = \frac{\pi}{2} K_{\mathrm{eff}}(T_{\mathrm{KT}}). \tag{283}$$

Therefore, if we can measure this stiffness, it should have a universal ratio to temperature at the point where it jumps to zero. This is seen experimentally, for instance in the value of the superfluid density for thin-films of Helium at the point of the transition from superfluid to normal liquid.

What does the RG flow tell us about the correlations in the XY model? Clearly, below T_{KT}, the system flows under renormalization to a critical fixed point with an algebraic correlation function, and an exponent equal to $T/2\pi K_{\rm eff}$. Above $T_{\rm KT}$, however, the system flows to a high-temperature fixed point where the correlation length becomes equal to the short-scale cut-off length. The actual correlation length must then be found by integrating back the flow equations, which we will now do: For $T > T_{\rm KT}$ we have the initial condition $x_0^2 - y_0^2 = -\chi^2 < 0$, where χ can be written,

$$\chi = \left(y_0^2 - x_0^2\right)^{1/2} = \left[b^2 e^{-2\beta\varepsilon_c} - (\pi\beta\varepsilon - 2)^2\right]^{1/2} \approx c\left[\frac{(T - T_{\rm KT})}{T_{\rm KT}}\right]^{1/2}, \qquad (284)$$

where the last result is the limiting form close enough to $T_{\rm KT}$. Now using the RG flow equation (277) we have,

$$\begin{aligned} dx &= -y^2 \frac{dl}{l}, \\ &= -(\chi^2 + x^2)\frac{dl}{l}, \\ \frac{dx}{x^2+\chi^2} &= -\frac{dl}{l}, \end{aligned} \qquad (285)$$

with solution,

$$\frac{1}{\chi}\tan^{-1}\frac{x}{\chi} - \frac{1}{\chi}\tan^{-1}\frac{x_0}{\chi} = \ln(l/l_0). \qquad (286)$$

Inverting this shows us how the length-scale l changes with the RG flow,

$$l = l_0 e^{-\frac{1}{\chi}(\tan^{-1}\frac{x}{\chi} - \tan^{-1}\frac{x_0}{\chi})}. \qquad (287)$$

Note that we expect x to flow to large positive values. However the $\tan^{-1}$ function will be limited to π as $x \gg \chi$, so that the limit of l will be

$$\xi(T) = l(x \to \infty) = l_0 e^{\pi/\chi} = l_0 e^{b\left[\frac{T_{\rm KT}}{(T-T_{\rm KT})}\right]^{1/2}}. \qquad (288)$$

Therefore we see an incredibly sudden divergence of the correlation length as the Kosterlitz-Thouless transition is approached from above. Note that while the exponent of 1/2 that appears is universal, the factor b, which also appears in the exponential, is non-universal, i.e. it depends on initial conditions.

We can now use a simple scaling argument to find the singular contribution to the free energy near the pahase transition. Assuming that the free-energy density scales as $f_s \sim \xi^{-d}$ we get,

$$f_s \propto e^{-bd\left[\frac{T_{\rm KT}}{(T-T_{\rm KT})}\right]^{1/2}}. \qquad (289)$$

(This result is backed up by a more careful RG analysis of the free energy.) Intriguingly, all orders of derivatives of this free energy are zero as $T \to T_{\rm KT}$. Therefore there are no discontinuities in any derivative of the free energy at the Kosterlitz-Thouless transition!

6.4 Physical examples

Apart from the quite beautiful theory, partially described above, of this Kosterlitz-Thouless phase transition, the interest in the problem also comes from its applicability to many important condensed matter systems. Apart from an easy-plane 2D ferromagnet, the same transition can occur in thin-film superfluids and superconductors (although in the latter the extra property of magnetic screening will kill the phase transition on very long lengths). Also, the dynamical phase transition of phase-slips in a very thin superconducting wire has been described as a KT transition using the quantum d-dimension to classical $(d+1)$-dimension mapping.

Other two-dimensional systems with a goldstone mode, such as crystals, can also be expected to have analagous topological defects to the vortex. For a crystal the relevant defect is a dislocation, and the theory describing unbinding of dislocation pairs is known as the Kosterlitz-Thouless-Halperin-Nelson-Young theory of 2D melting. The theory is more complicated as dislocations can only destroy some of the quasi-long-range order in a crystal: the system can retain orientational order, which is only destroyed if extra defects known as disclinations can proliferate. Whether there are two such transitions, of KT type, or if there is a single first-order transition, remains a question of controversy.

Finally, an interesting system arises when we have an external magnetic field perpendicular to a thin-film superconductor. The field forces a finite density of vortices (of same orientation) in to the film, and their mutual repulsion means that a lattice is formed at low enough temperatures. This system should therefore have both a unbinding transition of vortex–anti-vortex pairs, and an unbinding transition of dislocations in the vortex lattice. It turns out that the melting transition happens at a considerably lower temperature due to the smaller energy cost of a dislocation than a vortex (there is an extra factor of $1/4\pi$ in the prefactor of the logarithm!).