1 Lecture 8: Field-theoretic renormalization group. Low dimensional systems.

Here we shall describe another approach to renormalization based on representing fields as linear combinations of harmonics with different wavelengths, and summing over short wavelength fields. The general problem solved by the renormalization group theory is to understand the effect of fluctuations in a system with long range order. Thermodynamic fluctuations can be strong either near a critical point, where ordering emerges, or in the low temperature phase, in the case when the ordered state is described by a continuous symmetry group. In both cases, the space dimension has to be low enough for fluctuation effects to become significant. We shall describe the general RG framework, and then apply it first to a relatively more simple case of low dimensional systems with continuous symmetry, the main systems of interest being the nonlinear sigma model and the XY model. After gaining experience in renormalization techniques, we shall turn to the more difficult problem of fluctuations near a critical point.

1.1 RG in momentum space

In the most general terms, we are untested in a problem described by a partition function

$$Z = \mathrm{tr} e^{-\mathcal{H}(\phi)} \tag{1}$$

where $\mathcal{H}$ is a hamiltonian for some field ϕ describing ordering, such as spin density in a magnet, condensate wavefunction amplitude in a superfluid, etc.

The renormalization group transformation involves three standard steps: coarse-graining, rescaling, and hamiltonian renormalization.

(i) *Coarse-graining* is achieved by decomposing the field into a slow and a fast part, $\phi = \phi' + \delta\phi$. The fast and the slow fields are defined using Fourier representation as follows:

$$\phi(\mathbf{x}) = \sum_{|\mathbf{k}|<\Lambda'} e^{\mathbf{k}\dot{\mathbf{x}}}\phi_{\mathbf{k}} + \sum_{\Lambda'<|\mathbf{k}|<\Lambda} e^{\mathbf{k}\dot{\mathbf{x}}}\phi_{\mathbf{k}} \tag{2}$$

Here Λ is the ultraviolet cutoff, which is implicit in all problems of statistical mechanics: inverse lattice spacing, $\Lambda = 2\pi/a$, or interparticle distance, $\Lambda = 2\pi/d$. The intermediate cutoff $\Lambda' = \Lambda/b$ is of the order of inverse block size, in the real space RG picture. The slow field ϕ' is referred to as coarse-grained field.

(ii) *Rescaling* of space in the coarse-grained problem, $\mathbf{x}' = \mathbf{x}/b$, corresponds to rescaling of momentum $\mathbf{k} = b\mathbf{k}$. Upon rescaling, the cutoff Λ' is shifted back to Λ.

(iii) *Renormalized hamiltonian* is constructed so that the partition function remains invariant:

$$Z = \mathrm{tr}_{\phi'}\mathrm{tr}_{\delta\phi}\, e^{-\mathcal{H}(\phi'+\delta\phi)} = \mathrm{tr}_{\phi'}\, e^{-\mathcal{H}'(\phi')}, \tag{3}$$

which means that $\mathcal{H}'$ is defined by

$$e^{-\mathcal{H}'(\phi')} \equiv \langle e^{-\mathcal{H}(\phi'+\delta\phi)} \rangle_{\delta\phi} \tag{4}$$

(here $\langle ... \rangle_{\delta\phi}$ stands for $\mathrm{tr}_{\delta\phi} ...$).

In this formulation of RG procedure, the difficult part is calculating $\mathcal{H}'$ from $\mathcal{H}$. Typically, the trace in Eq.(4) can be evaluated only when the fast field statistics is gaussian. It is therefore useful to summarize here the basic facts about gaussian statistics.

1.2 Gaussian statistics

One variable distribution $\exp(-\frac{1}{2}A\phi^2)$ has a *characteristic function*

$$\langle e^{iJ\phi} \rangle = \exp(-\frac{1}{2}A^{-1}J^2) \tag{5}$$

This formula can be generalized for a distribution of a multicomponent variable ϕ_r of the form $\exp(-\frac{1}{2}\sum_{rr'} A_{rr'}\phi_r\phi_{r'})$. The characteristic function is

$$\langle e^{i\sum_r J_r\phi_r} \rangle = \exp(-\frac{1}{2}\sum J_r\tilde{A}_{rr'}j_{r'}) \tag{6}$$

where $\sum_{r''}\tilde{A}_{rr''}A_{r''r'} = \delta(r-r')$. For a continuum field variable $\phi(r)$ with a distribution $\exp(-\frac{1}{2}\int\int A(r,r')\phi(r)\phi(r')d^3rd^3r')$, this reads as

$$\langle e^{i\int J(r)\phi(r)d^3r} \rangle = \exp(-\frac{1}{2}\int\int \tilde{A}(r,r')J(r)J(r')d^3rd^3r') \tag{7}$$

with the inverse $\tilde{A}$ defined by $\int \tilde{A}(r,r'')A(r'',r')d^3r'' = \delta^{(3)}(r-r')$.

Pair correlation function[1] $\langle \phi(r_1)\phi(r_2) \rangle$ can be obtained by differentiating the characteristic function (7) with respect to $J(r_1)$ and $J(r_2)$, and then setting $J(r) = 0$. This gives an expression for the correlator in terms of $\tilde{A}$:

$$\langle \phi(r_1)\phi(r_2) \rangle = \tilde{A}(r_1, r_2) \tag{8}$$

From that, we obtain another useful formula

$$\langle e^{i\int J(r)\phi(r)d^3r} \rangle = \exp(-\frac{1}{2}\int\int \langle \phi(r)\phi(r') \rangle J(r)J(r')d^3rd^3r') \tag{9}$$

directly relating the distribution form with the pair correlation function.

In many cases of interest, $A(r,r')$ depends only on the coordinates difference $r-r'$. In such a case, the inverse $\tilde{A}$ has a simple form in the Fourier representation,

$$\tilde{A}(r-r') = \int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\tilde{A}(\mathbf{k})\frac{d^3k}{(2\pi)^3}. \tag{10}$$

[1] Here we do not distinguish between ordinary correlators, or moments, and the so-called irreducible correlators, or cummulants. Proper discussion of this will be given later.

Indeed, since the inverse $\tilde{A}(\mathbf{k}) = 1/A(\mathbf{k})$, we have

$$\langle \phi(r)\phi(r')\rangle = \tilde{A}(r-r') = \int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\frac{1}{A(\mathbf{k})}\frac{d^3k}{(2\pi)^3} \tag{11}$$

One can generalize these results to correlation functions of higher order, $\langle \phi(r)\phi(r')\phi(r'')...\rangle$, and derive the so-called Wick's theorem expressing the high order correlators in terms of pair correlators. We shall discuss it when time comes.

As an example of using the identity (11), let us consider the correlation function for fluctuations in Landau theory. These fluctuations, as we discussed in Lecture 4, are described by a quadratic Hamiltonian obtained by expanding Landau functional near its minimum:

$$\mathcal{H}(\delta m) = \int \frac{1}{2}K\left((\nabla\delta m)^2 + \xi^{-2}\delta m^2\right)d^dx \tag{12}$$

with $K\xi^{-2} \equiv \mathcal{H}''_L(m_0)$. The probability distribution of δm is gaussian, $\exp(-\beta\mathcal{H}(\delta m))$. Comparing with the above, we see that in this case A is a differential operator,

$$A(r,r') = \beta K\left(-\nabla^2 + \xi^{-2}\right) \tag{13}$$

To find $\tilde{A}$ we need to invert A. This is easily done in Fourier representation,

$$A(\mathbf{k}) = \beta K\left(\mathbf{k}^2 + \xi^{-2}\right), \quad \tilde{A}(\mathbf{k}) = \frac{T}{K\left(\mathbf{k}^2 + \xi^{-2}\right)} \tag{14}$$

which gives

$$\tilde{A}(r,r') = \int \frac{T}{K\left(\mathbf{k}^2 + \xi^{-2}\right)}\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\, \frac{d^dk}{(2\pi)^d} \tag{15}$$

This expression is identical to the result (14) of Lecture 4, obtained using the equipartition theorem for Fourier harmonics $\delta m_{\mathbf{k}}$.

1.3 Fluctuations in low dimensional systems with continuous symmetry

In low space dimension, thermal fluctuations suppress ordering or change it in a profound way. One example, discussed already in Lecture 6, is the absence of phase transitions in 1D Ising systems. Although we considered only a specific 1D model, the conclusion about the absence of ordering in 1D is very general. It applies to systems with all symmetries, discrete or continuous, as long as the interaction is short-range. However, in two-dimensional systems, as Onsager's problem illustrates, the Ising critical point is not destroyed by fluctuations.

The situation is different for systems with continuous symmetry in the ordered state. There is a famous theorem, due to Mermin and Wagner, and Hohenberg, stating that in such systems the long range order is destroyed by fluctuations at arbitrarily low temperature. The central example of systems with continuous symmetry, on which we focus below, is the nonlinear sigma model[2], describing fluctuations of a k-component unit vector field:

$$\mathcal{H} = \frac{1}{2g} \int \sum_{\mu=1,2;\, i=1,\dots,k} (\partial_\mu n_i)^2 d^2x \,, \quad |\mathbf{n}(x)| = \left(n_1^2(x) + \dots + n_k^2(x) \right)^{1/2} = 1 \tag{16}$$

with g a coupling constant. Although the Hamiltonian appears to be quadratic, the constraint $|\mathbf{n}| = 1$ makes it a nontrivial interacting field problem. The problem (16) possesses a continuous $SO(k)$ symmetry, its ground states are spatially uniform field configurations $\mathbf{n}(x) = \bar{\mathbf{n}}$, with $\bar{\mathbf{n}}$ an arbitrary unit vector on a $(k-1)$-dimensional sphere.

For $k = 2$, the problem is known as the XY model, describing 2D superfluids and planar ferromagnets with spin forced to lie in a plane. For $k = 3$, it is the Heisenberg ferromagnet problem, discussed in Lecture 1 (around Eq.20). For the Heisenberg problem on a lattice we derived the continual field problem (16) from a gradient expansion. The coupling constant in this case was found to be

$$g = T/J \tag{17}$$

with J the spin exchange parameter.

To see how the system (16) looses long range order due to fluctuations, we consider it at low temperature, $g = T/J \ll 1$, and perturb near a zero temperature state,

$$\mathbf{n}(x) = \mathbf{n}' \sqrt{1 - \mathbf{u}^2(x)} + \mathbf{u}(x) \,, \quad \mathbf{u} \cdot \mathbf{n}' = 0 \tag{18}$$

with constant $\mathbf{n}'$ and a vector $\mathbf{u}$ describing transverse fluctuations, $\mathbf{u} \perp \mathbf{n}'$. Assuming first that the fluctuations are small, we expand (16) in $\mathbf{u}$ up to quadratic order:

$$\mathcal{H}(\mathbf{u}) = \frac{1}{2g} \int \sum (\partial_\mu \mathbf{u})^2 d^2x = \frac{1}{2g} \int \sum (\partial_\mu u_i)^2 d^2x \,, \quad i = 1, \dots, k-1; \ \mu = 1, 2, \tag{19}$$

where u_i are coefficients of $\mathbf{u}$ expansion in an orthonormal basis $\mathbf{e}_i \perp \mathbf{n}'$.

Now let us consider the fluctuations of $\mathbf{u}$. Since the distribution $\exp(-\mathcal{H}(\mathbf{u}))$ is gaussian, we can use the above result (11) to evaluate

$$\langle u_i^2(x) \rangle = g \int \frac{1}{\mathbf{k}^2} \frac{d^2k}{(2\pi)^2} = \frac{g}{2\pi} \int_{2\pi/L}^{2\pi/a} \frac{dk}{k} = \frac{g}{2\pi} \ln L/a \tag{20}$$

Here we treat a logarithmically diverging integral by cutting it off at k larger than the inverse lattice spacing, and at k smaller than the inverse system size.

The result diverges for infinite system, when $L \to \infty$, indicating that our assumption about fluctuations being small is not consistent. Thus in an infinite system, the long range order at finite temperature, no matter how small, is destroyed by fluctuations. From

[2] A strange-sounding name is because this model, in a general field-theoretic form, appeared first in the high energy physics, in the context of theory of pion interaction.

the logarithmic expression (20), one can also conclude that the characteristic ordering persistence length beyond which correlations are suppressed by fluctuations is given by

$$\xi \simeq a \exp(2\pi/g) \tag{21}$$

The correlation length ξ remains finite at all temperatures, becoming exponentially large at small temperature $T/J = g \ll 1$.

Alternatively, one can consider fluctuations in an infinite system in the presence of en external field. Adding a term $-\int \mathbf{h}\cdot\mathbf{n}(x)d^2x$ to the hamiltonian, and using Eq.(18) to expand in fluctuations near the uniform state $\mathbf{n}(x) \parallel \mathbf{h}$, we obtain a hamiltonian with a mass term:

$$\mathcal{H}(\mathbf{u}) = \frac{1}{2g}\int \sum \left[(\partial_\mu \mathbf{u})^2 + g\beta|h|\mathbf{u}^2\right] d^2x \tag{22}$$

This quadratic hamiltonian defines a gaussian distribution of $\mathbf{u}(x)$. From (11), the variance is

$$\langle u_i^2(x)\rangle = g\int \frac{1}{\mathbf{k}^2 + g\beta|h|}\frac{d^2k}{(2\pi)^2} = \frac{g}{4\pi}\int_0^{(2\pi/a)^2} \frac{dw}{w + g\beta|h|} = \frac{g}{4\pi}\ln\frac{(2\pi)^2 J}{a^2|h|} \tag{23}$$

with $w = \mathbf{k}^2 + g\beta|h|$. We see that finite field h suppresses transverse fluctuations and, at small $T = gJ$, makes the state well ordered. However, in the limit $h \to 0$ the fluctuations diverge, indicating the absence of long range order at any finite temperature.

Comment: Somewhat unexpectedly, the short distance cutoff length a in this case is temperature dependent. Usually, the cutoff length is associated with lattice constant or interparticle spacing. In this case, however, due to Bose statistics of spin excitations, there is a cutoff length of a different origin. The equipartition theorem, which is essentially what Eq.(20) expresses, is valid only for the spin excitations $\omega(k) = Jk^2$ with sufficiently small frequency: $\hbar\omega(k) \le T$ (see Problem 2, PS#1). For higher frequency the Planck function $1/(e^{\beta\hbar\omega(k)} - 1)$ should be inserted in the integral in (20) instead of $g/|\mathbf{k}|^2$. This suppresses the contribution of the wavelengths shorter than $a = 2\pi(\hbar J/T)^{1/2}$. With the temperature dependent cutoff length a, the result (23) for transverse fluctuations takes the form $(T/4\pi J)\ln(T/|h|)$.

1.4 Renormalization of the nonlinear sigma model

It is interesting to take a closer look at how diverging fluctuations suppress long range order. For that we perform an RG analysis in the limit of small coupling g (i.e. low temperature), when the effect of divergnce is weakened and some form of perturbation theory can be used.

Generalizing the above analysis of transverse fluctuations, based on Eq.(18), we decompose the fluctuating order parameter as

$$\mathbf{n}(x) = \mathbf{n}'(x)\sqrt{1 - \mathbf{u}^2(x)} + \mathbf{u}(x)\,, \quad \mathbf{u}(x)\cdot\mathbf{n}'(x) = 0 \tag{24}$$

where $\mathbf{n}'(x)$ is a slowly varying part of $\mathbf{n}(x)$ and $\mathbf{u}(x)$ is the rapidly varying part. Fourier transformed, the field $\mathbf{u}(x)$ contains harmonics with $\Lambda' < k < \Lambda$, while $\mathbf{n}'(x)$ harmonics are $k < \Lambda'$. (Recall that in the real space RG Λ and Λ' are associated with inverse lattice constant $2\pi/a$ and inverse box size $2\pi/b$, respectively.)

We want to integrate the partition function over fast fluctuations $\mathbf{u}(x)$ and derive an effective hamiltonian $\mathcal{H}'$ for $\mathbf{n}'(x)$. To derive the coupling betweeen the slow and fast field, let us expand $\mathbf{u}(x)$ in an orthonormal basis of vectors $\mathbf{e}_i(x)$ orthogonal to $\mathbf{n}'(x)$:

$$\mathbf{u}(x) = u_i(x)\mathbf{e}_i(x)\,, \quad \mathbf{e}_i \cdot \mathbf{n}' = 0 \tag{25}$$

Using the basis $\mathbf{e}_i$, we define the coefficients $c_{\mu,i}$ as

$$\partial_\mu \mathbf{n}'(x) = c_{\mu,i}\mathbf{e}_i(x) \tag{26}$$

(note that the sum of squares gives $(c_{\mu,i})^2 = (\partial_\mu \mathbf{n}')^2$). Using the coefficients $c_{\mu,i}$, one can write $(\partial_\mu \mathbf{n})^2$ in the form

$$(\partial_\mu \mathbf{n})^2 = (\partial_\mu \mathbf{n}')^2(1-\mathbf{u}^2) + c_{\mu,i}c_{\mu,j}u_iu_j + (\partial_\mu \mathbf{u})^2 \tag{27}$$

which is well suited for averaging over fast field.

Let us comment on the approximation made in deriving the formula (27). To write the derivative $\partial_\mu \mathbf{u}$, with $\mathbf{u}$ expanded in the basis $\mathbf{e}_i$, we need to know the derivatives

$$\partial_\mu \mathbf{e}_i = -c_{\mu,i}\mathbf{n}' + f_{\mu,ij}\mathbf{e}_j \tag{28}$$

with $f_{\mu,ij}$ symmetric in i, j. Here the coefficients $c_{\mu,i}$ are the same as in the expression (28) (it follows from $\mathbf{e}_i \cdot \mathbf{n}' = 0$, $\mathbf{e}_i \cdot \mathbf{e}_j = 0$). By choosing position-dependent $\mathbf{e}_i(x)$, one can reduce all $f_{\mu,ij}$ to

$$f_{\mu,ij} = O(\partial^2 \mathbf{n}') \tag{29}$$

which sufficiently small to be safely ignored.

Using the result (27) we write the hamiltonian, separating the contributions of fast and slow fields:

$$\mathcal{H} = \frac{1}{2g}\int \Big((\partial_\mu \mathbf{n}')^2 + c_{\mu,i}c_{\mu,j}u_iu_j - (\partial_\mu \mathbf{n}')^2\mathbf{u}^2 + (\partial_\mu \mathbf{u})^2 \Big)\, d^2x \tag{30}$$

Since the field $\mathbf{u}$ has characteristic wavenumber $\Lambda' < k < \Lambda$, while $\mathbf{n}'$ has wavenumbers less than $\Lambda' \ll \Lambda$, last term, $(\partial_\mu \mathbf{u})^2$, is about $(\Lambda/\Lambda')^2$ times larger than the second and third terms. Because of that, as far as the fast field fluctuations are concerned, we can take the last term for the fast field hamiltonian,

$$\mathcal{H}_{\rm fast}(\mathbf{u}) = \frac{1}{2g}\int (\partial_\mu \mathbf{u})^2 d^2x \tag{31}$$

ignoring coupling to slowly varying field $\mathbf{n}'$. The effective hamiltonian for the slow field $\mathcal{H}'$ can then be obtained by averaging the $\mathbf{n}'$-dependent part of the hamilonian over $\mathbf{u}$,

$$\mathcal{H}'(\mathbf{n}') = \frac{1}{2g}\int \Big((\partial_\mu \mathbf{n}')^2 + \langle c_{\mu,i}c_{\mu,j}u_iu_j - (\partial_\mu \mathbf{n}')^2\mathbf{u}^2 \rangle_{\rm fast} \Big)\, d^2x \tag{32}$$

The average can be evaluated as follows:

$$\langle c_{\mu,i}c_{\mu,j}u_iu_j - (\partial_\mu \mathbf{n}')^2\mathbf{u}^2\rangle_{\text{fast}} = c_{\mu,i}c_{\mu,j}\langle u_iu_j\rangle - (\partial_\mu \mathbf{n}')^2\langle \mathbf{u}^2\rangle = -(k-1)\frac{g}{2\pi}\ln\frac{\Lambda}{\Lambda'}(\partial_\mu \mathbf{n}')^2 \quad (33)$$

where we used the equal point correlator

$$\langle u_iu_j\rangle = g\delta_{ij}\int_{\Lambda'}^{\Lambda}\frac{1}{\mathbf{k}^2}\frac{d^2k}{(2\pi)^2} = \frac{g}{2\pi}\delta_{ij}\ln\frac{\Lambda}{\Lambda'} \quad (34)$$

This resulting hamiltonian is

$$\mathcal{H}' = \int\left(\frac{1}{2g}(\partial_\mu \mathbf{n}')^2 - \frac{k-1}{4\pi}\ln\frac{\Lambda}{\Lambda'}(\partial_\mu \mathbf{n}')^2\right)d^2x \quad (35)$$

This means We see that the renormalized hamiltonian preserves its form, while the coupling is changed as $g \to g'$ with

$$\frac{1}{g'} = \frac{1}{g} - \frac{k-1}{2\pi}\ln\frac{\Lambda}{\Lambda'} \quad (36)$$

We can put this recursion relation in a differential form near the $g = 0$ fixed point. For that, choose the rescaling factor $\Lambda/\Lambda' \gg 1$ and simultaneously $g\ln(\Lambda/\Lambda') \ll 1$. In this case, the change of coupling after one renormalization step is relatively small, $\delta g = g' - g \ll g$, and one can write

$$\delta g = \frac{k-1}{2\pi}g^2\ln\frac{\Lambda}{\Lambda'} \quad (37)$$

In terms of the RG "time" variable $t \equiv \ln(\Lambda_0/\Lambda)$, with $\Lambda_0 = a^{-1}$, we have a differential RG equation

$$\frac{dg}{dt} = \frac{k-1}{2\pi}g^2 \quad (38)$$

The solution of this equation,

$$\frac{1}{g(t)} = \frac{1}{g_0} - \frac{k-1}{2\pi}t, \quad (39)$$

defines the RG flow of the coupling. (Note that the differential equation reproduces the above recursion relation obtained for a finite change of t.) The result of this analysis is that the effective coupling increases with the length scale changed from a to a' as

$$g(a') = \frac{g_0}{1 - \frac{k-1}{2\pi}g_0\ln(a'/a)} \quad (40)$$

Although in deriving this equation we assumed $g \ll 1$, and thus, striktly speaking, we cannot extrapolate the result to $g \gg 1$, it is suggestive that the system becomes disordered when the length scale $a' = ae^{2\pi/(k-1)g_0}$ is reached. This gives an estimate for the correlation length:

$$\xi \simeq a\,\exp\frac{2\pi J}{(k-1)T} \quad (41)$$

which refines the estimate drawn from the Mermin-Wagner theorem.

The disorder induced by thermal fluctuations is manifest in the absence of phase transition. Experimentally, transverse fluctuations are not directly accessible, but it turns out that magnetization curve $m(h)$ measured in the presence of external field, has certain qualitative features directly related to the anomalous behavior of transverse fluctuations. First, $m \to 0$ as $h \to 0$, since there is no long range order. (Recall that $m(h=0) > 0$ in a ferromagnetic state with spontaneous magnetization.) Also, the dependence m vs h at low temperature has a logarithmic dependence on the field h:

$$\delta m = m_{T=0} - m_{T>0} = 1 - \langle (1-\mathbf{u}^2)^{1/2} \rangle_{T>0} = \frac{1}{2}\langle \mathbf{u}^2 \rangle_{T>0} = k\frac{T}{8\pi J}\ln\frac{T}{h} \tag{42}$$

which is correct when the suppression of magnetization by fluctuations is small, $\delta m \ll 1$, i.e. at not too low field. At $h \to 0$, the dependence $m(h)$ is linear. The susceptibility $\chi = dm/dh|_{h=0}$ can be related to the correlation length (see Problem 2, PS#6).

1.5 Summarize

- **Field-theoretic renormalization** is based on coarse-graining in Fourier space. The field harmonics are separated into slow and fast, $k < \Lambda'$ and $\Lambda' < k < \Lambda$, with an intermediate cutoff Λ' playing the same role as the block size b in the real space RG (in fact, $\Lambda' \sim b^{-1}$). The RG transformation of a hamiltonian is performed by summing over fast fields in the partition function, and representing the result as a partition function generated by an effective hamiltonian.

- **Long range order with conitnuous symmetry** is destroyed by thermal fluctuations in space dimension $d \leq 2$. This can be seen either from a perturbative analysis of the transverse fluctuations divergence, or from a more systematic field-theoretic RG. The RG flow of coupling constant (inverse temperature) drives the system from weak to strong coupling, i.e. to high effective temperature, indicating the absence of long range order.