1 Lecture 6: Real space RG. Blocking, decimation, bonds-moving.

1.1 Renormalization by decimation. 1D Ising model

There are several ways to implement real space renormalization. The real space RG is a flexible approach: there is a large freedom in choosing block spins, as well as in making approximations when the block Hamiltonian is derived. We demostrate the main ideas in several simple examples. We will work with a minimal set of couplings (one-spin and two-spin) and ignore more complicated couplings that are typically generated by the RG procedure. Although the real space RG accuracy can be systematically improved by enlarging the coupling parameter space, we will not pursue this here[1].

Let us consdier a 1D Ising problem

$$\mathcal{H} = -K\sum_i s_i s_{i+1}\,, \quad K = J/T\,, \quad s_i = \pm 1 \tag{1}$$

and carry out *a decimation* in which all the spins on the odd sites $i = 2n+1$ are summed over, and the partition function is re-expressed in terms of the spins on the even sites $i = 2n$ (Fig.1.1).

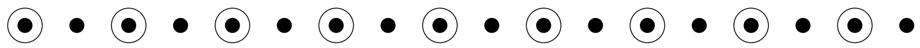

Figure 1: Decimation RG for 1D Ising model. The spins on one family of sites are summed over to obtain an effective Hamiltonian for remaining sites.

Let us transform the partition function:

$$Z = \sum_{s_i=\pm 1}\prod_i e^{K(s_i s_{i+1}} = \sum_{s_{2n}}\prod_{i=2n}\left(\sum_{s_{2n+1}} e^{K(s_{2n}s_{2n+1}+s_{2n+1}s_{2n+2})}\right) \tag{2}$$

We factored Z to separate the contributions of the odd sites. To sum over s_{2n+1} in (2), we rewrite

$$e^{Ks_i s_{i+1}} = \cosh K\,(1 + us_i s_{i+1})\,, \quad u = \tanh K \tag{3}$$

Then,

$$\sum_{s_{2n+1}} e^{J(s_{2n}s_{2n+1}+s_{2n+1}s_{2n+2})} = \cosh^2 K\sum_{s_{2n+1}}(1+us_{2n}s_{2n+1})(1+us_{2n+1}s_{2n+2}) \tag{4}$$

$$= 2\cosh^2 K(1+u^2 s_{2n}s_{2n+2}) \tag{5}$$

[1] We refer to the article by Th. Niemeijer and J. M. J. van Leeuwen, Phys. Rev. Lett., v. 31, p.1411 (1973). This is a classic paper, in which the the numerical power of real space RG and the possibility to achieve quantitative accuracy was first demonstrated for the Ising model on a 2D triangular lattice.

Thus, up to a rescaling factor $2\cosh^2 K$, the partition function form is reproduced with $u \to u^2$. Going back to K, we have

$$\tanh K' = \tanh^2 K \tag{6}$$

which can be rewritten as a recursion relation:

$$K' = \frac{1}{2} \ln \cosh(2K) \tag{7}$$

The decimated problem describes spins on a lattice with the spacing two times the original spacing. The interaction is of nearest neighbor form, as before, with the effective (or, renormalized) interaction constant given by K'.

The renormalization procedure can now be repeated many times, giving

$$K'' = \frac{1}{2} \ln \cosh(2K'), \quad K''' = \frac{1}{2} \ln \cosh(2K''), \quad ... \tag{8}$$

The qualitative picture of the coupling flow defined by the RG transformation (7) is quite simple: since $K' < K$ for any finite $K > 0$, the only two fixed points are $K = 0, \infty$. This is shown schematically in Fig. 1.1

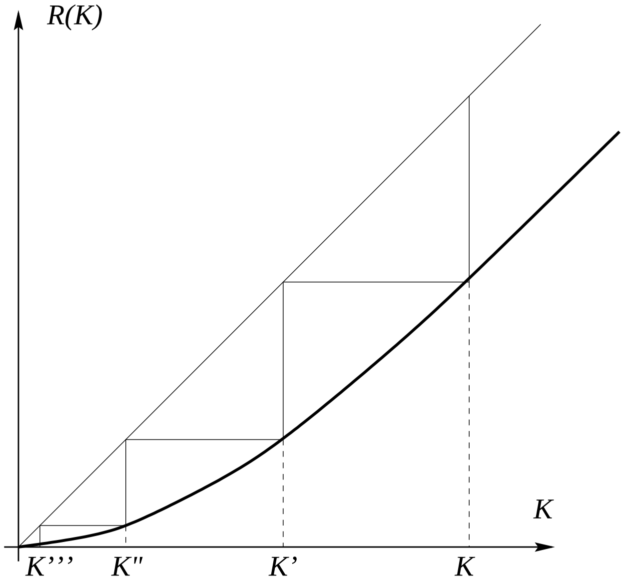

Figure 2: RG equation for 1D Ising model analyzed grafically. Coupling flows from large values to small, indicating a disordered state at arbitrarily low temperature.

The point $K = 0$ is an attraction point of the RG transformation ($K' = K^2$ for small K). Physically, small $K = J/T$ corresponds to weak coupling or high temperature. The other fixed point, $K = \infty$, is unstable. Indeed, at large K the recursion relation (7) takes the form

$$K' = K - \frac{\ln 2}{2} \tag{9}$$

and so, even if the starting value of K is large, after several renormalization steps it is reduced to $K \simeq 1$. Large $K = J/T$ corresponds to strong coupling or low temperature. The conclusion, therefore, is that at some length scale the coupling strength always becomes small, no matter what the microscopic coupling was. Thus there is *no long range order* in this system. The fluctuations in $D = 1$ totally eliminate the phase transtion. At large length scales, the low temperature state is the same as at high temperature, i.e. totally disordered.

The absence of phase transition can also be seen from partition function analyticity in the coupling constant K. The partition function for this problem is easily evaluated exactly. For that, we introduce *link variables* defined as $\xi_i = s_i s_{i+1}$. The Hamiltonian, written in terms of ξ_i, is a sum of independent terms, one per each link:

$$\mathcal{H} = -\sum_i K\xi_i\,, \quad \xi_i = \pm 1 \tag{10}$$

One can also note that the original spin variables s_i can be expressed through the link variables as

$$s_i = s_0 \prod_{j<i} \xi_j \tag{11}$$

where s_0 is the leftmost spin. (We assume that the chain is finite, $i = 0, ..., N$.) This means that the summation over $s_i = \pm 1$ in the partition function can be replaced by the summation over $\xi_i = \pm 1$ and $s_0 = \pm 1$. This gives

$$Z = \mathrm{tr}\,_s e^{-\mathcal{H}} = \sum_{s_0=\pm 1} \prod_i \left(\sum_{\xi_i=\pm 1} e^{K\xi_i} \right) = 2^{N+1} \left(\cosh K\right)^N \tag{12}$$

Then the free energy per site, $F = -T \ln(2\cosh K)$, is an analitic function, and so all thermodynamics quantities are also analytic in temperature. This confoirms our conclusion, drawn from the RG flow properties, about the absence of long range order and phase transition.

Let us discuss the situation at low temperature $T \ll J$. In this case, neighboring spins are exponentially more likely to be aligned than antialigned: $p(s_i = s_{i+1}) \simeq e^{2J/T} p(s_i \neq s_{i+1})$. Hence, there is a high degree of short-range order and locally the system looks well ordered. However, at distant points, there is no correlation in spin alignment (no macroscopic ordering). The correlation length within which the ordering persists can be estimated roughly as the length scale at which the probability to find antialigned spins is of order one: $\xi \simeq e^{2J/T}$

Another way to estimate the correlation length ξ is to use the strong coupling limit of the RG recursion relation (9). The number of decimation steps after which the coupling $K \gg 1$ is reduced to $K \sim 1$ is

$$n \approx \frac{2}{\ln 2} K \tag{13}$$

This defines $\xi = 2^n = e^{2K} = e^{2J/T}$, the same length scale as above.

1.2 Blocking by moving bonds

In space dimension higher than one, there is usually no possibility to carry out the renormalization exactly, like in the above 1D example. Thus various approximations need to be made. Historically, one of the first successful schemes was proposed by Migdal in his work on lattice gauge theories, as an attempt to solve quantum chromodynamics on a lattice. This approach is based on a very simple idea which we describe below and then apply to the 2D Ising model.

Suppose one wants to carry out the decimation procedure on a 2D square lattice with couplings along the links. There are two sublattices, even and odd, so a natural thing to do would be to sum over spins on one of the sublattices and obtain effective couplings for the remaining spins. One can see that after the first decimation step, in additin to the nearest neighbor couplings, we generate next nearest neghbor couplings. Thus the nearest neighbor Hamiltonian does not preserve its form.

We face a dilemma: either neglect the new couplings by setting them to zero after each RG step, or add more couplings to keep track of. In fact, none of these alternatives is perfect. Neglecting the next nearest neighbor coupling altogether gives a recursion relation identical to the one obtained in the 1D Ising model, and thus no phase transition. Adding new couplings will generate even more couplings at subsequent RG steps, and we will have to employ some truncation procedure to reign the exploding couplings.

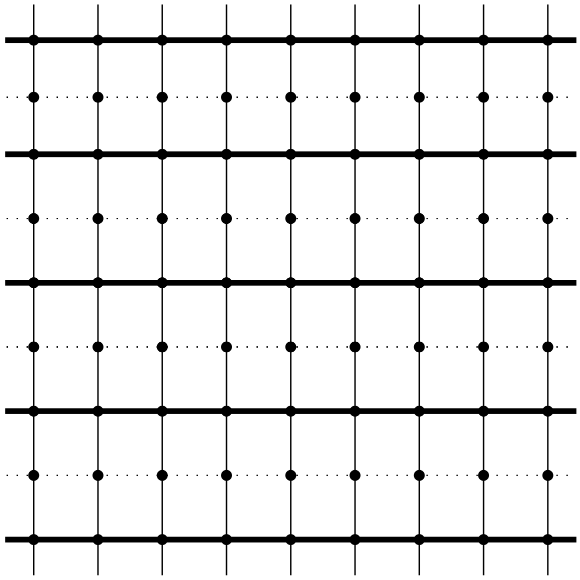

Figure 3: Bond-moving scheme illustrated. The couplings on the bonds shown by dotted lines are set to zero. At the same time, the couplings on the thick bonds are doubled to account for the reduction in other couplings.

The bond-moving scheme, in contrast, handles the couplings more gently. The unwanted couplings are not discarded, but moved elsewhere, so that the resulting structure is more amenable to decimation. We start with the anisotropic Ising model with two different couplings K_x and K_y. We divide all spins into two groups, according to whether their *row number* is even or odd (Fig. 1.2). Our first step is to shift the x couplings from odd columns to neighboring even colums: $K_{x,2i+1} \to 0$, $K_{x,2i} \to 2K_{y,2i}$. After that, the spins in odd rows can be easily summed over in the partition function. These spins are coupled only with their neighbors in the y direction, and the decimation is carried in the same way as in the 1D example. Therefore, the RG transformation has the form

$$K'_y = \frac{1}{2} \ln \cosh 2K_y \,, \quad K'_x = 2K_x \tag{14}$$

If this procedure is repeated again and again, K_y will decrease, while K_x will grow. To prevent this unphysical anisotropy build up, instead of simply repeating the procedure at the next step, we interchange the columns and rows. That means that now we shift the y couplings from odd columns to even, and then decimate by summing over spins in the odd columns. Then

$$K''_y = 2K'_y \,, \quad K''_x = \frac{1}{2} \ln \cosh 2K'_x \tag{15}$$

After that, we keep interchaging rows and columns at every RG step.

Let us focus on the recursion relation for the y couplings. After two RG steps,

$$K''_y = \ln \cosh 2K_y \tag{16}$$

The flow defined by this equation can be studied grafically (Fig. 4). Besides the two familiar fixed points $K = 0, \infty$, there apears one more fixed point with finite $K = K_*$. This fixed point is unstable. For initial coupling strength $K < K_*$, the RG flow makes K even weaker, whereas $K > K_*$ grow and flow to the $K = \infty$ fixed point, which is now a stable fixed point.

The interpretation of this behavior is that there are two stable fixed points, $K = 0, \infty$, correspond to the disordered and ordered states, while the unstable fixed point $K = K_*$ describes the vicinity of the phase transition with $T_c = J/K_*$.

Discuss generalization to higher space dimension. In a cubic lattice of dimension d, at each RG step, the number of bonds to be moved is $2(d-1)$ for each site, with 1D-like decimation in the remaining direction:

$$K'_x = \frac{1}{2} \ln \cosh 2K_x \,, \quad K'_y = 2K_y \,, \quad K'_z = 2K_z \,, ... \tag{17}$$

The direction in which the bonds are preserved is changed at each step to maintain cubic symmetry on average. The fixed point and its qualitative properties are similar to the 2D case.

Despite being an *ad hoc* procedure, the bond-moving RG scheme gives surpisingly good results for a wide range of problems. However, it introduces anisotropy by breaking the $x - y$ symmetry of the Ising problem. Also, it is hard to modify this approach to improve the accuracy. Because of that, this approach is usually used only for illustrative purposes. Many other renormalization schemes have been developed that do not break symmetries and allow for systematic accuracy control. However, since our purpose here is mainly to build the intuition about renormalization approach, we will not discuss these alternative schemes.

Figure 4:

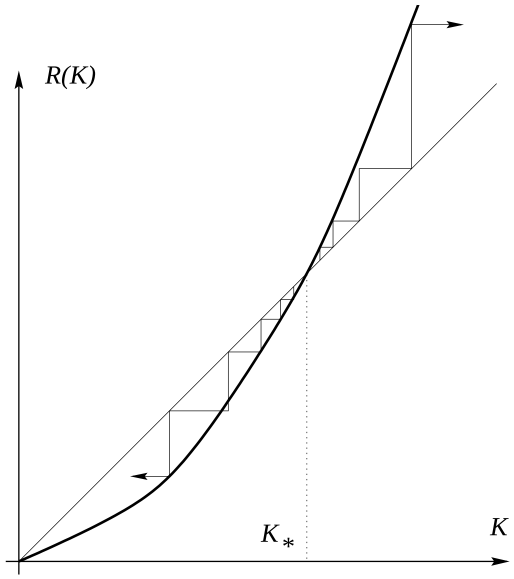

Figure 5: RG equation for 2D Ising model analyzed grafically. The unstable fixed point corresponds to the phase transition. Depending on the initial conditions (coupling strength or temperature) the flow takes the system to $K = 0$ (disordered state) or to $K = \infty$ (ordered state).

1.3 Summarize

- **Real space renormalization** is an intuitive and simple mathematical procedure, in which coarse-graining is performed on Hamiltonians defined on a lattice. The RG transformation is exact in the space of all Hamiltonians containinhg both the couplings present in the microscopic Hamiltonian and the couplings of more complex form generated via blocking.

- **Different types of ordering** correspond to different stable fixed points of the RG transformation. Each stable fixed point is a point of attraction of the RG flow, with the limiting Hamiltonian describing the macroscopic properties of a coarse-grained system.

- **Phase transitions** correspond to unstable fixed points of the RG transformation. By varying microscopic parameters, e.g. temperature or coupling strength, one can change initial conditions of the RG flow. Near an unstable fixed point, a small change in the initial conditions may correspond to a very large change in macroscopic behavior, i.e. to a different thermodynamic ordering phase.