1 Lecture 12: Field-theoretic perturbative RG. Wilson-Fisher fixed point.

Example of a nontrivial fixed point born in a vicinity of a trivial fixed point:

$$\frac{du}{dl} = ku - au^2 \tag{1}$$

Two fixed points: $u = 0$ and $u = u_* = k/a$.

Eigenvalues, scaling dimension of u. Perturbation theory in small k: From RG flow for some coupling u_1,

$$\frac{du_1}{dl} = y_1 u_1 + buu_1 + O(u^2, u_1^2) \tag{2}$$

one can estimate the sacling dimesnion of u_1 at the u_* fixed point:

$$y_1^* = y_1 + bu_* + O(u_*^2) \tag{3}$$

From RG flow for u_1 in the vicinity of $u = o$ known as series in u up to order n one can predict scaling dimension at the nontrivial fixed point up to $O(u_*^{n+1})$.

1.1 Coarse-graining. Cummulants. Wick's theorem.

Landau hamiltonian

$$\mathcal{H} = \int \left[\frac{1}{2}\left(\tau\phi^2 + K(\nabla\phi)^2\right) + u\left((\phi^2)^2\right)\right] d^d x \tag{4}$$

for an n-component field $\phi_i(\mathbf{x})$.

Coarse graining in momentum space,

$$\phi_i(\mathbf{x}) = \int e^{i\mathbf{q}\cdot\mathbf{x}} \phi_i(\mathbf{q}) d^d x \tag{5}$$

$$\phi_i(\mathbf{x}) = \phi_{i,<}(\mathbf{x}) + \phi_{i,>}(\mathbf{x}) \tag{6}$$

with decomposition into $\phi_<$ and $\phi_>$ defined using shells in momentum space:

$$\phi(\mathbf{q}) = \begin{cases} \phi_<(\mathbf{q}) & 0 < |\mathbf{q}| < \Lambda' \\ \phi_>(\mathbf{q}) & \Lambda' < |\mathbf{q}| < \Lambda \end{cases} \tag{7}$$

where Λ is of order of inverse lattice spacing and the intermediate cutoff Λ' is related with the rescaling factor b as $\Lambda' = \Lambda/b$.

To coarse-grain the problem, let us first separate the hamiltonian (4) into a quadratic and a quartic part, $\mathcal{H} = \mathcal{H}_0 + U$, where

$$\mathcal{H}_0 = \int \frac{1}{2}\left(\tau + K\mathbf{q}^2\right)|\phi_{\mathbf{q}}|^2 \frac{d^d q}{(2\pi)^d} \tag{8}$$

$$U = \iiiint u\left(\phi_{i,\mathbf{q}_1}\phi_{i,\mathbf{q}_2}\right)\left(\phi_{j,\mathbf{q}_3}\phi_{j,\mathbf{q}_4}\right)(2\pi)^d\delta(\mathbf{q}_1 + ... + \mathbf{q}_4)\frac{d^d q_1 d^d q_2 d^d q_3 d^d q_4}{(2\pi)^{4d}} \tag{9}$$

Now separate the summation over $\phi_<$ and $\phi_>$ in the partition function:

$$Z = \mathrm{tr}_<\mathrm{tr}_> e^{-\mathcal{H}_0(\phi_< + \phi_>)}e^{-U(\phi_< + \phi_>)} = \mathrm{tr}_< e^{-\mathcal{H}_0(\phi_<)}\mathrm{tr}_> e^{-\mathcal{H}_0(\phi_>)}e^{-U(\phi_< + \phi_>)} \tag{10}$$

$$= Z_{0,>}\mathrm{tr}_< e^{-\mathcal{H}_0(\phi_<)+\delta\mathcal{H}(\phi_<)} \tag{11}$$

with constant prefactor $Z_{0,>} = \mathrm{tr}_> e^{-\mathcal{H}_0(\phi_>)}$ and correction to the hamiltonian $\delta\mathcal{H}$ defined by

$$\delta\mathcal{H}(\phi_<) = -\ln\langle e^{-U(\phi_< + \phi_>)}\rangle_{\mathcal{H}_0} \tag{12}$$

Here $\langle ...\rangle_{\mathcal{H}_0}$ means averaging over gaussian distribution $P(\phi_>) \propto e^{-\mathcal{H}_0(\phi_>)}$. To compute $\delta\mathcal{H}$ use cummulant expansion

$$\ln\langle e^{-U}\rangle_{\mathcal{H}_0} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\langle\!\langle U^n\rangle\!\rangle \tag{13}$$

$$\langle\!\langle U\rangle\!\rangle = \langle U\rangle\,, \quad \langle\!\langle U^2\rangle\!\rangle = \langle (U - \langle U\rangle)^2\rangle\,, \quad \langle\!\langle U^3\rangle\!\rangle = \langle (U - \langle U\rangle)^3\rangle\,, \quad ... \tag{14}$$

The cummulants, aka irreducible moments, are of interest not only here but in general, since they are extensive quantities (i.e. scale with system volume). This is not true for ordinary moments $\langle U^n\rangle$.

In computing cummulants of $U(\phi_< + \phi_>)$ we can benefit from gaussian character of the distribution $e^{-\mathcal{H}_0(\phi_>)}$ and use the general rule known as *Wick's theorem*. For correlators of Fourier harmonics $\phi_{i,\mathbf{q}}$ of order one, two, three, and four one can show that

$$\langle\phi_{i,\mathbf{q}}\rangle = 0\,, \quad \langle\phi_{i,\mathbf{q}}\phi_{i',\mathbf{q}'}\phi_{i'',\mathbf{q}''}\rangle = 0 \tag{15}$$

$$\langle\phi_{i,\mathbf{q}}\phi_{i',\mathbf{q}'}\rangle = \frac{\delta_{ii'}\delta(\mathbf{q} + \mathbf{q}')}{\tau + K\mathbf{q}^2}, \tag{16}$$

$$\begin{aligned}\langle\phi_{i_1,\mathbf{q}_1}\phi_{i_2,\mathbf{q}_2}\phi_{i_3,\mathbf{q}_3}\phi_{i_4,\mathbf{q}_4}\rangle &= \langle\phi_{i_1,\mathbf{q}_1}\phi_{i_2,\mathbf{q}_2}\rangle\langle\phi_{i_3,\mathbf{q}_3}\phi_{i_4,\mathbf{q}_4}\rangle + \langle\phi_{i_1,\mathbf{q}_1}\phi_{i_3,\mathbf{q}_3}\rangle\langle\phi_{i_2,\mathbf{q}_2}\phi_{i_4,\mathbf{q}_4}\rangle \\ &+ \langle\phi_{i_1,\mathbf{q}_1}\phi_{i_4,\mathbf{q}_4}\rangle\langle\phi_{i_3,\mathbf{q}_3}\phi_{i_2,\mathbf{q}_2}\rangle\end{aligned} \tag{17}$$

The generalization for an arbitrary correlator $\langle\phi_{i_1,\mathbf{q}_1}...\phi_{i_n,\mathbf{q}_n}\rangle$ is as follows. All correlators with an odd n are zero, while the correlators with even n are given by

$$\langle\phi_{i_1,\mathbf{q}_1}...\phi_{i_n,\mathbf{q}_n}\rangle = \sum_{\text{all pair contractions}}\langle\phi_1\phi_2\rangle\langle\phi_3\phi_4\rangle...\langle\phi_{n-1}\phi_n\rangle \tag{18}$$

defined by breaking the n fields $\phi_{i_1,\mathbf{q}_1}$, ... $\phi_{i_n,\mathbf{q}_n}$ into pairs in all possible ways. Each pair contributes the pair correlator (16), independent of other pairs. The number of terms in the sum is equal to the number of ways to divide n onjects into pairs, equal to

$$(n-1)(n-3)...3\cdot 1 = n!/2^{n/2}(n/2)! \tag{19}$$

1.2 Perturbative RG

Armed with the Wick's theorem, we are ready to face the cummulants $\langle\langle U^n(\phi_< + \phi_>)\rangle\rangle_{\mathcal{H}_0}$. Since our aim is modest – to go to the first order in $\epsilon = 4 - d$ in the RG, we only need to know $\langle U \rangle$ and $\langle\langle U^2 \rangle\rangle$. By using a brut force approach, or something more refined, such as a graphical representation (Feynman diagrams), we obtain

$$\langle U(\phi_< + \phi_>)\rangle_{\mathcal{H}_0} = U(\phi_<) + (4+2n)G^{(1)}_{\Lambda,\Lambda'} u \sum_{|\mathbf{q}|<\Lambda'} \phi_{i,\mathbf{q}}\phi_{i,-\mathbf{q}} \tag{20}$$

where

$$G^{(1)}_{\Lambda,\Lambda'} = \sum_{\Lambda'<|\mathbf{p}|<\Lambda} \frac{1}{\tau + K\mathbf{p}^2} \tag{21}$$

The graphs enumerating various contributions to $\langle U \rangle$ are shown in Fig.1.2.

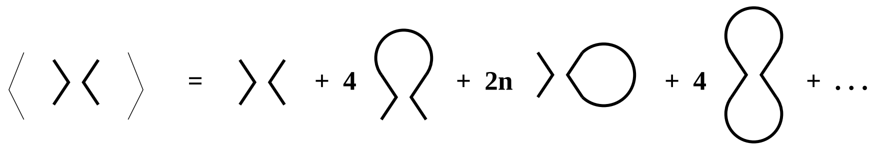

Figure 1: Graphs for $\langle U \rangle$ are shown. The first term is $u(\phi_<^2)^2$, the second and third are $G^{(1)}_{\Lambda,\Lambda'} u\phi_<^2$ times combinatorial factors, other terms are constants.

The 4-leg vertex $\rangle\langle$ represents the interaction $(\phi^2)^2$. The legs, representing field ϕ are joined for the fields from the same dot product. Averaging is performed by pairing together fast fields, using Wick's theorem. Free ends represent slow field, arcs joining the ends represent pair average of fast fields, given by Eq.(21). Graphs with the same topology are pictured by one diagram. Combinatorial factors are given by the number of ways to form specific graph by pairing fast fields together.

Tensor summation is performed over indices of the fields joined by a line or a loop, thus each line with free ends has a Kronecker delta $\delta_{ii'}$ associated with it, contracted with slow fields at the ends. Each closed loop contributes $\sum_i \delta_{ii} = n$.

Momentum conservation $\delta(\mathbf{q}_1 + ... + \mathbf{q}_4)$ at each vertex, and for pairs of fast fields in the average (16) by Wick's rule, ensure that the sum of all momenta of fields at free ends is zero. The momenta in the inner parts of the graphs take values $\Lambda' < |\mathbf{q}| < \Lambda$, while the momenta at the ends are constrained by $|\mathbf{q}| < \Lambda'$.

Similarly, for the second cummulant we have

$$\langle\langle U^2(\phi_< + \phi_>)\rangle\rangle_{\mathcal{H}_0} = (4n+48)u^2 \sum_{\mathbf{q}_1+...+\mathbf{q}_4=0} G^{(2)}_{\Lambda,\Lambda'}(\mathbf{q}_1+\mathbf{q}_2)\phi_{i,\mathbf{q}_1}\phi_{i,\mathbf{q}_2}\phi_{j,\mathbf{q}_3}\phi_{j,\mathbf{q}_4} \tag{22}$$

plus terms quadratic in ϕ which we will not need. Here

$$G^{(2)}_{\Lambda,\Lambda'}(\mathbf{q}) = \sum_{\Lambda'<|\mathbf{p}|<\Lambda} \frac{1}{(\tau + K\mathbf{p}^2)(\tau + K(\mathbf{p}+\mathbf{q})^2)} \tag{23}$$

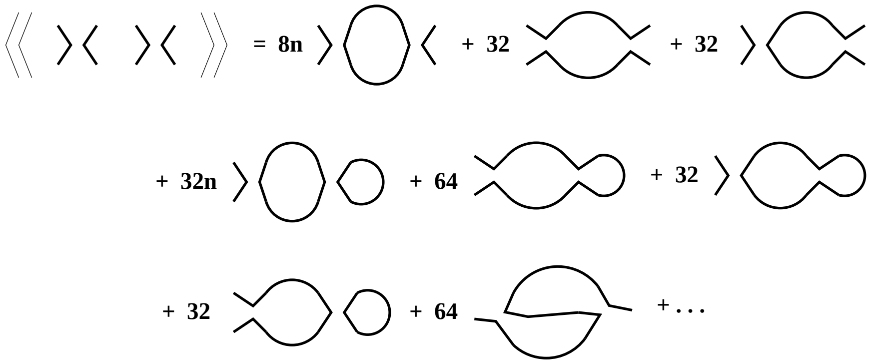

Figure 2: Graphs for $\langle\langle U^2 \rangle\rangle$ are shown. The first, second and third graphs contribute to the $u^2(\phi_<^2)^2$ term, other graphs represent contributions of $\phi_<^2$ form.

Graphically, these contributions are represented in Fig.1.2.

This gives coarse-grained hamiltonian to the order $O(u^3)$:

$$\mathcal{H} + \delta\mathcal{H} = \int_0^{\Lambda'} \left(\frac{1}{2}(\tau + K\mathbf{q}^2) + u(2n+4)G^{(1)}_{\Lambda,\Lambda'} - \frac{1}{2}u^2 A(\tau, \mathbf{q}) \right) |\phi_\mathbf{q}|^2 \frac{d^d q}{(2\pi)^d} \tag{24}$$

$$+ \int_0^{\Lambda'} ... \int_0^{\Lambda'} \left(u - \frac{1}{2}u^2(8n+64)G^{(2)}_{\Lambda,\Lambda'}(0) \right) (2\pi)^d \delta_{\mathbf{q}_1+...+\mathbf{q}_4} (\phi_{i,\mathbf{q}_1}\phi_{i,\mathbf{q}_2})(\phi_{j,\mathbf{q}_3}\phi_{j,\mathbf{q}_4}) \frac{d^d q_1 ... d^d q_4}{(2\pi)^{4d}}$$

$$+ O(u^3) \tag{25}$$

Here $A(\tau, \mathbf{q})$ is obtained from the ϕ^2 contributions depicted in Fig1.2. In what follows we will not need to know their exact form.

From the coarse-grained hamiltonian, we obtain recursion relations:

$$\tilde{K} = K - u^2 A''(0) \tag{26}$$

$$\tilde{\tau} = \tau + 4(n+2)uG^{(1)}_{\Lambda,\Lambda'} - u^2 A(0) \tag{27}$$

$$\tilde{u} = u - 4(n+8)u^2 G^{(2)}_{\Lambda,\Lambda'}(0) \tag{28}$$

The next step is to rescale momentum,

$$\mathbf{q} = b^{-1}\mathbf{q}' \tag{29}$$

which shifts Λ' back to Λ, and to renormalize field,

$$\phi' = z\phi \tag{30}$$

After that the parameters in $\mathcal{H}$ change as

$$K' = b^{-d-2}z^2\tilde{K}\,, \quad \tau' = b^{-d}z^2\tilde{\tau}\,, \quad u' = b^{-3d}z^4\tilde{u} \tag{31}$$

Now let us fix the renormalization factor z so that $K' = K$. This gives

$$z = b^{(d+2)/2}\left(1 + O(u^2)\right) \tag{32}$$

To analyze the RG relations, it is conveniant to put them in a differential form. We define the RG "time" parameter as $l = \ln(\Lambda/\Lambda')$ and obtain

$$\frac{d\tau}{dl} = 2\tau + \frac{4u(n+2)K_d\Lambda^d}{\tau + K\Lambda^2} - Au^2 \tag{33}$$

$$\frac{du}{dl} = (4-d)u - \frac{4(n+8)K_d\Lambda^d}{(\tau + K\Lambda^2)^2}u^2 \tag{34}$$

The second equation has the form we anticipated, with the trivial fixed point $u = 0$ loosing stability at $d < 4$, and simultaneously a new fixed point $u = u_*$ born nearby.

To the fiorst order in $\epsilon = 4 - d$, the fixed point is

$$u_* = \frac{K^2}{4(n+8)K_4}\epsilon + O(\epsilon^2) \tag{35}$$

$$\tau_* = -\frac{n+2}{2(n+8)}K\Lambda^2\epsilon + O(\epsilon^2) \tag{36}$$

Linearizing RG flow at the fixed point (u_*, τ_*), obtain eigenvalues

$$y_t = 2 - \frac{n+2}{n+8}\epsilon + O(\epsilon^2) \tag{37}$$

$$y_u = -\epsilon + O(\epsilon^2) \tag{38}$$

Nagative y_u indicates stability of the new fixed point.

The properties of physical quantities can be analyzed using the relation between thermal eigenvalue y_t and scaling of thermodynamical esatblished earlier in the context of real space RG. For example, the correlation length scaling behavior is $\xi \sim (\delta\tau)^{-\nu}$, with

$$\nu = \frac{1}{y_t} = \frac{1}{2} + \frac{n+2}{4(n+8)}\epsilon + O(\epsilon^2) \tag{39}$$

which corrects the mean field result $\nu = 1/2$.

For specific heat, $C \sim \tau^{-\alpha}$, we have

$$\alpha = 2 - d\nu = 2 - \frac{4-\epsilon}{2}\left(1 + \frac{n+2}{2(n+8)}\epsilon + O(\epsilon^2)\right) = \frac{4-n}{2(n+8)}\epsilon + O(\epsilon^2) \tag{40}$$

Compare with experimental values for the Ising ($n = 1$), XY ($n = 2$), and Heisenberg ($n = 1$) cases.

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