4.6 Assumption of local equiibrium

We have thus observed that it is possible to have generic power law scaling of correlations in non-equilibrium systems with conservation laws and anisotropy. Not-surprisingly, given the long-range correlations, the presence of boundaries modifies correlations. These are the conditions we identified earlier as ingredients for long-range FIF. The question thus arises if we can identify FIF in this class of systems.

The issue, however, is how to relate fluctuations to force and energy. In case of thermal equilibrium, this was achieved by converting the probability distribution of fluctuations to an energy using a Boltzmann weight, i.e. using $U = -k_B T \ln \mathcal{P}[\phi]$ with an appropriate temperature with which the fluctuating system is in equilibrium. We do not have this luxury for out of equilibrium circumstances, and need an alternative approach.

A heuristic approach is to again to a nearby equilibrium state. For example, let us imagine that $\phi(\mathbf{x}, t)$ describes non-equilibrium fluctuations of a density, i.e. $\rho(\mathbf{x}, t) = \overline{\rho} + \phi(\mathbf{x}, t)$. Further assume that the nearby equilibrium system is endowed with a pressure (or stress tensor) related to density $\overline{\rho}$ that can provide an measure of force. The *assumption of local equilibrium* is that out of equilibrium there is a local time dependent pressure (or stress) evaluated at $\rho(\mathbf{x}, t) = \overline{\rho} + \phi(\mathbf{x}, t)$, i.e.

$$P(\mathbf{x}, t) = P(\overline{\rho} + \phi(\mathbf{x}, t)) = P(\overline{\rho}) + P'(\overline{\rho})\phi(\mathbf{x}, t) + \frac{1}{2}P''(\overline{\rho})\phi(\mathbf{x}, t)^2 + \cdots . \tag{4.28}$$

Averaging the above equation then leads to

$$P(\mathbf{x}, t) = P(\overline{\rho} + \phi(\mathbf{x}, t)) = P(\overline{\rho}) + \frac{1}{2}P''(\overline{\rho})\langle \phi(\mathbf{x}, t)^2 \rangle_{non.\ eq.} + \cdots . \tag{4.29}$$

The subscript on the last term is to emphasize that the additional contribution to pressure comes from non-equilibrium fluctuations captured by ϕ, and not the equilibrium fluctuations in density.

4.7 Long-ranged FIF in a diffusive current

Since the application of equilibrium pressure to a non-equilibrium situation is questionable, its validity certainly needs to be checked in various settings. We recently [2] studied a somewhat related (but not identical) problem of non-equilibrium FIF for particles diffusing from a dense to a dilute reservoir. The set-up considered is depicted in Fig. 4.1.

Simulations were conducts on a lattice in which particles randomly moved from site to site. The resulting density varies along the x-direction, such that

$$\langle \rho(x, y) \rangle = \rho_l + (\rho_r - \rho_l)\frac{x}{L}. \tag{4.30}$$

In the absence of current, the pressure of the lattice gas is calculated easily and equals

$$P(\rho) = -\frac{k_B T}{a^2}\ln(1-\rho), \quad \Longrightarrow \quad P"(\rho) = \frac{k_B T}{a^2}\frac{1}{(1-\rho)^2}. \tag{4.31}$$

[2] A. Aminov, Y. Kafri, and M. Kardar, Phys. Rev. Lett. **114**, 230602 (2015).

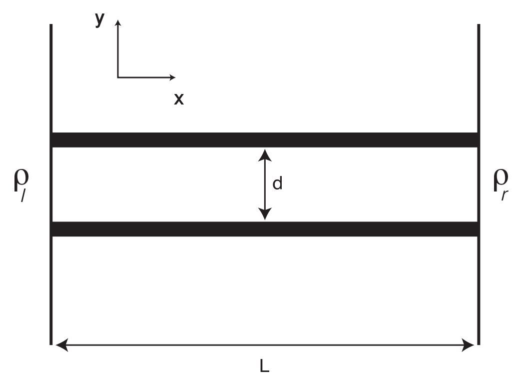

Fig. 4.1 The setups studied is a two dimensional system, infinite in the y direction and connected to two reservoirs at $x = 0$ and $x = L$. Two slabs, a distance d from each other, span the system along the x direction. The two reservoirs have densities $\rho(0, y) = \rho_l$ and $\rho(L, y) = \rho_r$.

There are of course fluctuations in density even in the absence of current, but the presence of the current leads to additional non-equilibrium density fluctuations which can be computed by methods similar to those used above. It is important to note that there are density fluctuations both from the large volumes above and below the enclosed interval, as well as from within the interval. In fact fluctuations from the inside are smaller by a factor inversely proportional to the area Ld, leading to an attraction between the plates. The difference between the two non-equilibrium density fluctuations gives

$$\delta\langle\phi(\mathbf{x}, t)^2\rangle_{non.\ eq.} = -a^2 \frac{(\rho_l - \rho_r)^2}{Ld} \frac{x}{L} \left(1 - \frac{x}{L}\right) , \tag{4.32}$$

Note that in this system there is no translational symmetry. Both the density, and its fluctuations vary from one reservoir to the other.

Given the expression for $P"(\rho)$, the assumption of local equilibrium thus predicts a non-equilibrium Casimir pressure that varies with x as

$$P_{non.\ eq.}(x) = -\frac{1}{2} \frac{k_B T}{(1 - \langle\rho(x)\rangle)^2} \frac{(\rho_l - \rho_r)^2}{Ld} \frac{x}{L} \left(1 - \frac{x}{L}\right) . \tag{4.33}$$

This prediction was tested numerically and the results, as presented in Fig. 4.2 are in excellent agreement.

4.8 Transient long-ranged FIF

Yet another way to generate non-equilibrium fluctuations is to agitate a system with conserved quantities. Assuming that our system is described by the most generic form described earlier, including isotropy, agitation corresponds to increasing the strength of noise. As discussed earlier, at the linear level, the generic isotropic system is equivalent to model B dynamics of a local Hamiltonian, and will eventually settle to a steady

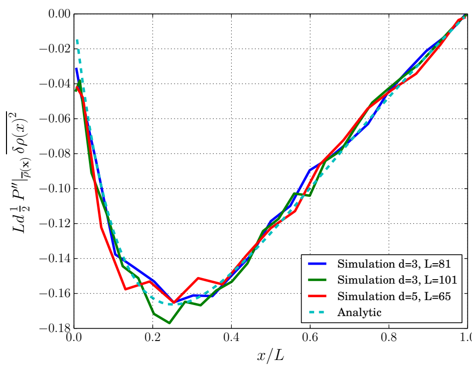

Fig. 4.2 Numerical results for the fluctuation induced pressure on one of the slabs for a system with $L = 101$, $d = 3$, $\rho_L = 0.1$ and $\rho_R = 0.7$. The units are chosen such that the lattice spacing is set to be $a = 1$ and $k_B T = 1$. The solid line depicts the numerical result and the dashed line represents the analytic prediction.

state with only short-range spatial correlations. However, due to the conservation law relaxation to the new steady state takes a very long (macroscopic) time.

In the linearized model, the Fourier amplitudes evolve as

$$\tilde{\phi}(\mathbf{q}, t) = e^{-\mu q^2 t}\tilde{\phi}(\mathbf{q}, 0) + \int_0^t dt' e^{-\mu q^2 (t-t')}\tilde{\eta}(\mathbf{q}, t)\,. \tag{4.34}$$

Each discrete mode thus evolves independently with zero mean and variance

$$\langle|\tilde{\phi}(\mathbf{q}, t)|^2\rangle = e^{-2\mu q^2 t}\langle|\tilde{\phi}(\mathbf{q}, 0)|^2\rangle + \int_0^t dt_1' dt_2' e^{-\mu q^2(2t-t_1'-t_1')} \overbrace{\langle|\tilde{\eta}(\mathbf{q}, t)|^2\rangle}^{2D_F q^2 \delta(t_1'-t_2')}\ . \tag{4.35}$$

Let us denote $\langle|\tilde{\phi}(\mathbf{q}, 0)|^2\rangle = T_I$ for the initial level of fluctuations, and $T_F = D_F/\mu = langle|\tilde{\phi}(\mathbf{q}, \infty)|^2\rangle$ for the final steady state. Then

$$\langle|\tilde{\phi}(\mathbf{q}, t)|^2\rangle = T_F + (T_I - T_F)e^{-2\mu q^2 t}\,. \tag{4.36}$$

Before the agitated system settles to its new steady state, transient correlations in space behave as

$$\langle\phi(\mathbf{x}, t)\phi(\mathbf{0}, t)\rangle = \int \frac{d^d q}{(2\pi)^d} e^{i\mathbf{q}.\mathbf{x}}\langle|\tilde{\phi}(\mathbf{q}, t)|^2\rangle \tag{4.37}$$

$$= T_F \delta^d(\mathbf{x}) + (T_I - T_F)\int \frac{d^d q}{(2\pi)^d} e^{i\mathbf{q}.\mathbf{x}-2\mu q^2 t} \tag{4.38}$$

$$= T_F \delta^d(\mathbf{x}) + (T_I - T_F)\exp(-\frac{x^2}{8\mu t})\left(\frac{1}{8\pi\mu d}\right)^{d/2}\ . \tag{4.39}$$

The correlations rise slowly following the initiation of the "quench" at $t \to 0$, reaching a maximum (when $x^2/(8\mu t_m^2) - d/(2t_m) = 0$) at $t_m = x^2/(4d\mu)$. At their maximum, the correlations behave as

$$\langle \phi(\mathbf{x}, t)\phi(\mathbf{0}, t)\rangle_{max} = \frac{T_I - T_F}{|\mathbf{x}|^d} \frac{e^{-d/2}}{(4\pi)^{d/2}}. \tag{4.40}$$

At their respective maxima, the correlations fall of as $|\mathbf{x}|^{-d}$, the same exponent encountered earlier. At very long times, the correlations decay as $\ell(t)^{-d}$, were $\ell(t) = \sqrt{8\pi\mu t}$ is the diffusive length. As in previous examples, we expect that these transient long-range correlations give rise to transient FIF, a proposition that we explored in recent studies [3].

[3] C. M. Rohwer, M. Kardar, and M. Krüger, Phys. Rev. Lett. **118**, 015702 (2017); C. M. Rohwer, A. Solon, M. Kardar, and M. Krüger, Phys. Rev. E **97**, 032125 (2018).