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Fluctuation Induced Forces (FIF) out of Equilibrium

4.1 Dissipative dynamics in open systems

In many contexts in nature, such as drifting clouds or flowing rivers, we can identify some interesting quantity (such as density) which varies over time, to all intents and purposes stochastically. Fluctuations are partially the result of the open nature of the system, with constant excahnges of particles, energy and other constituents with the 'environment.' It is not likely that an approach to the dynamics of such phenomena, starting from fundamental principles has any hope of success. Here, we instead take a general approach to following the dynamics of open and extended systems that is similar in spirit to the Landau's construction of effective coarse-grained field theories in equilibrium. Let us again consider the dynamics of a static field, $\phi(\mathbf{x}, t)$:

1. The starting point in equilibrium statistical mechanics is the Hamiltonian $\mathcal{H}[\phi]$. Landau's prescription is to include in $\mathcal{H}$ *all terms consistent with the symmetries* of the problem. The underlying philosophy is that in a generic situation an allowed term is present, and can only vanish by accident. In the case of non-equilibrium dynamics we shall assume that the *equation of motion* is the fundamental object of interest. Over sufficiently long time scales, inertial terms ($\propto \partial_t^2\phi$) are irrelevant in the presence of dissipative dynamics, and the evolution of h is governed by

$$\partial_t \phi(\mathbf{x}, t) = \overbrace{v\left[\phi(\mathbf{x}, t)\right]}^{\text{deterministic}} + \overbrace{\eta(\mathbf{x}, t)}^{\text{stochastic}} \quad . \tag{4.1}$$

2. If the interactions are short ranged, the velocity at $(\mathbf{x}, t)$ depends only on $\phi(\mathbf{x}, t)$ and a few derivatives evaluated at $(\mathbf{x}, t)$, i.e.

$$v(\mathbf{x}, t) = v\big(\phi(\mathbf{x}, t), \nabla\phi(\mathbf{x}, t), \cdots\big). \tag{4.2}$$

3. We must next specify the functional form of deterministic velocity, and the correlations in noise. Generalizing Landau's prescription, we assume that all terms consistent with the underlying *symmetries and conservation rules* will generically appear in v. The noise, $\eta(\mathbf{x}, t)$, may be conservative or non-conservative depending on whether there are only internal rearrangements, or external inputs and outputs.

(Note that with these set of rules there is no reason for the velocity to be derivable from a potential ($v \neq -\hat{\mu}\delta\mathcal{H}/\delta\phi$), and there is no fluctuation–dissipation connection. It is even possible for the deterministic velocity to be conservative, while the noise is not. Thus various familiar results of near equilibrium dynamics may no longer hold.)

A generic expansion of $v[\phi]$ assuming locality, small fluctuations, as well as spatial variations, takes the form

$$v[\phi] = -a\phi + b\phi^2 + \cdots + \frac{K}{2}\nabla^2\phi + \frac{K}{2}\nabla^4\phi + \cdots . \tag{4.3}$$

The various terms in the above expansion should be generically present, unless explicitly forbidden by symmetry or some other principle. The leading term in the above expansion is $-a\phi$, and generically imparts a characteristic time scale $\tau = a^{-1}$ for relaxation of fluctuations. There must be an explicit reason for relaxation of fluctuations to occur over macroscopic time scales as opposed to a microscopic τ.

• One possibility is the tuning of an external parameter (e.g. environment temperature T) on which system parameters depend to arrive at the special point $a(T_c) = 0$. This is similar to what happens at a critical point in an equilibrium setting, but the need to tune makes this a non-generic mechanism.

• A more generic situation is caused by a continuous symmetry, requiring the dynamics to be the same under the transformation $\phi(\mathbf{x}) \to \phi(\mathbf{x}) + \epsilon$. The requirement $v[\phi(\mathbf{x}) + \epsilon]$, then forbids any dependence on ϕ. We discussed such symmetry already in the context of the phonons (Goldstone) modes of a superfluid.

• A more interesting case, specific to dynamics is the presence of a conserved quantity as discussed next.

4.2 Conservation laws

As an example consider the flow of water along a river (or traffic along a highway). The deterministic part of the dynamics is conservative (the amount of water, or the number of cars is unchanged), requiring

$$\frac{d}{dt}\int d^dx\, \phi(\mathbf{x}, t) = \int d^dx \frac{\partial\phi(\mathbf{x}, t)}{\partial t} = 0. \tag{4.4}$$

This equation is satisfied if $\partial_t\phi(\mathbf{x}, t)$ is the divergence of a current, i.e. $-\nabla\vec{j}\,[\phi]$, with $\vec{j} = \vec{j}_D + \vec{j}_S$ including deterministic and stochastic components. The deterministic current, $\vec{j}_D$, is a vector, and must be constructed out of the other vectorial quantities in the problem: the gradient operator ∇ provides the only such operator in an isotropic system. The leading term in the expansion now starts with the Laplacian operator,

$$\partial_t\phi\mathbf{x}, t) = \mu\nabla^2\phi + \cdots , \tag{4.5}$$

and equivalently in terms of Fourier modes

$$\partial_t\tilde{\phi}(\mathbf{q}, t) = -\mu q^2\tilde{\phi}(\mathbf{q}, t) + \cdots , \tag{4.6}$$

The characteristic relaxation time of fluctuations is now related to macroscopic length scales ℓ, as $\tau(\ell) \sim \mu\ell^2$. (Higher order terms can potentially change the scaling exponent, but are unlikely to remove the dependence of relaxation time on macroscopic scales.)

It is not strictly necessary in an open system for the stochastic noise to be conservative. One can imagine cases, as in a traffic along a highway, in which the deterministic flow is conservative, with stochastic changes as cars leave and enter through exits. However, in the following we shall assume that the conservation condition is strict, applying also to the stochastic noise. In its most generic form, conservative white noise has zero mean, with covariance

$$\langle \eta(\mathbf{x}, t)\eta(\mathbf{x}', t') = 2D\delta(t - t')\nabla^2\delta^d(\mathbf{x} - \mathbf{x}') + \cdots, \tag{4.7}$$

which in Fourier space reads

$$\langle \tilde{\eta}(\mathbf{q}, t)\tilde{\eta}(\mathbf{q}', t') = \left[2Dq^2 + \cdots\right] \delta(t - t')(2\pi)^d\delta^d(\mathbf{q} + \mathbf{q}'). \tag{4.8}$$

4.3 Scale invariance?

Does the absence of a microscopic time scale imply the absence of a microscopic length scale? Let us examine the generic conservative linear model in Fourier space

$$\partial_t\tilde{\phi}(\mathbf{q}, t) = -\mu q^2\tilde{\phi}(\mathbf{q}, t) + \cdots, \tag{4.9}$$

whose solution in time (assuming $\phi(\mathbf{x}, t = 0) = 0$ for simplicity) satisfies

$$\tilde{\phi}(\mathbf{q}, t) = \int_0^t dt' e^{-\mu q^2(t-t')}\tilde{\eta}(\mathbf{q}, t). \tag{4.10}$$

The mean at all times is zero, while the covariances satisfy

$$\langle \tilde{\phi}(\mathbf{q}, t)\phi(\mathbf{q}', t)\rangle = \int_0^t dt_1' dt_2' e^{-\mu q^2(t-t_1')-\mu q^2(t-t_2')}\langle \tilde{\eta}(\mathbf{q}, t)\tilde{\eta}(\mathbf{q}', t)\rangle. \tag{4.11}$$

Using the co-variance of noise from Eq. (4.8) leads to

$$\langle \tilde{\phi}(\mathbf{q}, t)\phi(\mathbf{q}', t)\rangle = (2\pi)^d\delta^d(\mathbf{q} + \mathbf{q}')(2Dq^2) \int_0^t dt_1 e^{-2\mu q^2(t-t')}. \tag{4.12}$$

At long times $t \to \infty$ the integral over time equals $(2\mu q^2)^{-1}$, leading to the simple result

$$\langle \tilde{\phi}(\mathbf{q}, t)\phi(\mathbf{q}', t)\rangle = (2\pi)^d\delta^d(\mathbf{q} + \mathbf{q}')\frac{D}{\mu}. \tag{4.13}$$

The q-independent noise leads to long-time (steady state) correlations in real space of

$$\langle \phi(\mathbf{x}, t)\phi(\mathbf{x}', t)\rangle = \int \frac{d^dq}{(2\pi)^d}\frac{d^dq'}{(2\pi)^d} e^{i\mathbf{q}.\mathbf{x}+i\mathbf{q}'.\mathbf{x}'}\langle \tilde{\phi}(\mathbf{q}, t)\tilde{\phi}(\mathbf{q}', t)\rangle \tag{4.14}$$

$$= \int \frac{d^dq}{(2\pi)^d} e^{i\mathbf{q}.\mathbf{x}-i\mathbf{q}.\mathbf{x}'}\frac{D}{\mu} = \frac{D}{\mu}\delta^d(\mathbf{x} - \mathbf{x}'). \tag{4.15}$$

Thus, despite the fact that relaxation times are macroscopic, in this model the correlations are short-ranged.

The above calculation presents an example of the more general case of so called model B dynamics. Starting with a functional $\mathcal{H}[\phi]$, a dynamics can be constructed as

$$\partial_t \phi = -\nabla.(\vec{j}_D + \vec{j}_S), \quad \text{with} \quad \vec{j}_D = -\mu\nabla\frac{\delta\mathcal{H}}{\delta\phi}, \quad \text{and} \quad \vec{j}_S = \sqrt{2D}\nabla(\text{white noise}). \tag{4.16}$$

Such stochastic dynamics for a field $\phi(\mathbf{x}, t)$, subject to a conservation law on $\int d^d x\phi(\mathbf{x}, t)$, leads at long times to an equilibrium steady state

$$\mathcal{P}[\phi] \propto \exp\left(-\beta\mathcal{H}[\phi]\right), \qquad \text{with} \qquad \beta = \frac{\mu}{D}. \tag{4.17}$$

Although we had not imposed such a requirement by hand, our attempts to construct a low order linear theory on the basis of isotropy and conservation lead to dynamics similar to model B with the local Hamiltonian $\mathcal{H}[\phi] = \int d^d x\phi(\mathbf{x})^2/2$.

4.4 Generic scale invariance from anisotropy

Grinstein, Sachdev, and Lee [1] pointed out that anisotropy is sufficient to generically generate long-range spatial correlations in the previously considered linear model. Assume that due to some asymmetry (e.g. flow or a gradient of some field) different directions of space are not equivalent, with a specific direction (labeled parallel) distinguished in some manner. Due to such anisotropy the generic forms of relaxation and noise coefficients in Fourier space are modified from the isotropic case as

$$\mu q^2 \to \mu_\perp q_\perp^2 + \mu_\parallel q_\parallel^2, \qquad \text{and} \qquad Dq^2 \to D_\perp q_\perp^2 + D_\parallel q_\parallel^2 . \tag{4.18}$$

(Somewhat schematically, this can be thought of as introducing two distinct temperatures, $D_\parallel/\mu_\parallel$ and $D_\perp/\mu_\perp$ along the different directions.) In this case,

$$\langle|\tilde{\phi}_{\mathbf{q}}|^2\rangle = \frac{D_\perp q_\perp^2 + D_\parallel q_\parallel^2}{\mu_\perp q_\perp^2 + \mu_\parallel q_\parallel^2} , \tag{4.19}$$

is no longer q-independent, leading to long-range spatial correlations.

To quantify consequences of anisotropy on spatial correlations, let us simplify by assuming that only the noise is anisotropic, $D_\perp \neq D_\parallel$, while $\mu_\perp = D_\parallel = \mu$. To obtain correlations in real space, we now need integrals of the form

$$\int \frac{d^d q}{(2\pi)^d} e^{i\mathbf{q}.\mathbf{x}} \frac{q_\alpha q_\beta}{q^2} = -\partial_\alpha\partial_\beta \int \frac{d^d q}{(2\pi)^d} e^{i\mathbf{q}.\mathbf{x}} \frac{q_\alpha q_\beta}{q^2} \tag{4.20}$$

$$= \partial_\alpha\partial_\beta \left(\frac{|\mathbf{x}|^{2-d}}{(2-d)S_d}\right) = \frac{1}{S_d}\partial_\alpha \left(\frac{x_\beta}{|\mathbf{x}|^d}\right) \tag{4.21}$$

$$= \frac{1}{S_d}\left(\frac{\delta_{\alpha\beta}}{|\mathbf{x}|^d} - d\frac{x_\alpha x_\beta}{|\mathbf{x}|^{d+2}}\right). \tag{4.22}$$

This leads to

$$\langle\phi(\mathbf{x})\phi(\mathbf{0})\rangle = \frac{1}{\mu S_d}\int \frac{d^d q}{(2\pi)^d} e^{i\mathbf{q}.\mathbf{x}} \frac{D_\perp q_\perp^2 + D_\parallel q_\parallel^2}{q^2} \tag{4.23}$$

[1] G. Grinstein, S. Sachdev, and D.H. Lee, Phys. Rev. Lett. **64**, 1927 (1990).

$$= \frac{1}{\mu S_d}\left[D_\perp\left(\frac{d-1}{|\mathbf{x}|^d} - d\frac{x_\perp^2}{|\mathbf{x}|^{d+2}}\right) + D_\parallel\left(\frac{1}{|\mathbf{x}|^d} - d\frac{x_\parallel^2}{|\mathbf{x}|^{d+2}}\right)\right] \tag{4.24}$$

$$= \frac{1}{\mu S_d}\left[\frac{(d-1)D_\perp + D_\parallel}{|\mathbf{x}|^d} - d\frac{D_\perp x_\perp^2 + D_\parallel x_\parallel^2}{|\mathbf{x}|^{d+2}}\right]. \tag{4.25}$$

Thus, unless $D_\perp = D_\parallel$, the correlations are long-ranged, falling off as $|\mathbf{x}|^{-d}$, and cannot be expressed as originating from the Boltzmann weight of a local Hamiltonian.

More generally, with relaxation $\mu(\mathbf{q})$ and noise correlation $D(\mathbf{q})$, a generic linear model leads to $\langle|\tilde{\phi}_\mathbf{q}|^2\rangle = D(\mathbf{q})/\mu(\mathbf{q})$. Only if this ratio can be written as a analytic series in q^2 (i.e. if $D(\mathbf{q})/\mu(\mathbf{q}) = a + bq^2 + \cdots$), can we express the resulting weight as coming from the familiar Boltzmann weights of a local Hamiltonian. In the absence of the *detailed balance* symmetry of equilibrium we thus expect long-range correlations to be generic in noisy anisotropic systems with conserved quantities.

4.5 Correlations in confined geometry

Now consider confining the field $\phi(\mathbf{x}, t)$ in the Casimir set-up, between two plates at separation H. For simplicity, we further impose Neumann ("no flux") boundary conditions, which render all points statistically similar. For such a boundary condition, the Fourier modes are discretized with wavevectors $q_n = \pi n/H$ for $n = 0, 1, 2\cdots$ (note the inclusion of the $n = 0$ mode in this case.)

The variance of the field at any point is now obtained as

$$\langle\phi(\mathbf{x})^2\rangle = \frac{1}{V}\sum_\mathbf{q}\langle|\tilde{\phi}_\mathbf{q}|^2\rangle. \tag{4.26}$$

Note the $1/V$ factor, which ensures that $\langle\phi(\mathbf{x})^2\rangle$ remains an intensive quantity. In the 'isotropic' case with $\langle|\tilde{\phi}_\mathbf{q}|^2\rangle = D/\mu$, this leads to $\langle\phi(\mathbf{x})^2\rangle = (D/\mu)(N/V)$, where N is the total number of modes in the system after appropriate discretization. In general, we treat the wave-number $\mathbf{q}$ as continuous in the directions parallel to the confining plates, and discrete in the orthogonal direction, leading to

$$\langle\phi(\mathbf{x})^2\rangle = \frac{1}{H}\sum_{n=0}\int\frac{d^{d-1}q_\parallel}{(2\pi)^{(d-1)}}\,\frac{D_\parallel q_\parallel^2 + D_\perp(n\pi/H)^2}{\mu q_\parallel^2 + \mu(n\pi/H)^2}. \tag{4.27}$$

Note that the above integrals are somewhat similar to those encountered in computations of the thermal Casimir effect, as in Eq. (1.53) (for $d = 2$) and Eq. (1.62) (for $d = 3$). (With the exception of the term $n = 0$, the integrals become in fact identical to the above equations up to a factor of $D_\perp/\mu$ for $D_\parallel = 0$.) We thus expect the expression for $\langle\phi(\mathbf{x})^2\rangle$ to include a term decaying as $1/H^d$, proportional to $(D_\perp - D_\parallel)/\mu$ with a universal coefficient. Thus, due to the long-range nature of correlations in the non-equilibrium setting, the internal correlations become sensitive to the presence of the boundaries, acquiring corrections (which can be of either sign, depending on $(D_\perp - D_\parallel)$) that scale as H^{-d}.